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Regularity of the American Put option in the Black–Scholes model with general discrete dividends

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Abstract

We analyze the regularity of the value function and of the optimal exercise boundary of the American Put option when the underlying asset pays a discrete dividend at known times during the lifetime of the option. The ex-dividend asset price process is assumed to follow the Black–Scholes dynamics and the dividend amount is a deterministic function of the ex-dividend asset price just before the dividend date. This function is assumed to be non-negative, non-decreasing and with growth rate not greater than 1. We prove that the exercise boundary is continuous and that the smooth contact property holds for the value function at any time but the dividend dates. We thus extend and generalize the results obtained in Jourdain and Vellekoop (2011) [10] when the dividend function is also positive and concave. Lastly, we give conditions on the dividend function ensuring that the exercise boundary is locally monotonic in a neighborhood of the corresponding dividend date.

MSC: 60G40; 91A60; 91G20

Keywords: Optimal stopping; American options; Dividends; Early exercise boundary; Smooth contact property

0. Introduction

We consider the American Put option with maturity T and strike K written on an underlying stock S. Like in [10], we assume that the stochastic dynamics of the ex-dividend price process

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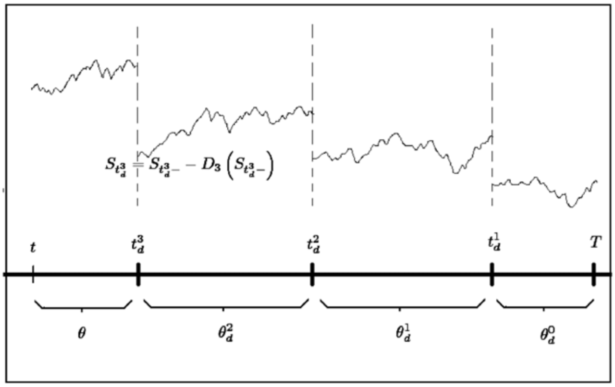


Fig. 1. A trajectory of the stock price process.

of this stock can be modeled by the Black–Scholes model and that this stock is paying discrete dividends at deterministic times $0 \leq t_d^I < t_d^{I-1} < \cdots < t_d^i < \cdots < t_d^1 < T$. At each dividend time t_d^i, the value of the stock becomes $S_{t_d^i} = S_{t_d^i-} - D_i(S_{t_d^i-})$ where $D_i(S_{t_d^i-})$ is the value of the dividend payment (see Fig. 1). We suppose that each dividend function $D_i : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, non-negative and such that $x \mapsto x - D_i(x)$ is also non-decreasing and non-negative.

We are interested in the value of the American Put option with strike K and maturity T. Since we are in a Markovian framework, the price can be characterized in terms of a value function depending of the time t and the stock price at time t. For the sake of consistency, we will denote this value function by u_0 for the case without dividends.

By change of numeraire, the pricing problem of the American Put option in the Black–Scholes model with continuously paid proportional dividends is equivalent to the pricing problem of the American Call option obtained by exchange of the spot value of the underlying and the strike and exchange of the dividend and interest rates. The latter problem was studied in [15] by McKean who first linked this optimal stopping-time problem to a free boundary problem involving both the value function and the exercise boundary. Van Moerbeke [20] derived an integral equation which involves both the exercise boundary and its derivative. Kim [11] later obtained an integral equation which only involves the exercise boundary itself. Independently, Jacka [9] and Carr et al. [6] derived the analogue equation for the exercise boundary c_0 of the American Put option in the Black–Scholes model without dividends. According to Jacka [9], the boundary c_0 is continuous, the first time the price process crosses c_0 is an optimal stopping time and the smooth fit property holds for the value function u_0. The uniqueness for the integral equation was left as an open problem in those papers. Uniqueness was proven by Peskir [16]. We refer to [17, Section. 25.] for a more recent exposition of these results. Convexity of c_0 was proved in [5] and in [7]. Infinite regularity of c_0 at all points prior to the maturity was formally proved by Chen and Chadam [4]. Then Bayraktar and Xing [2] proved that this remains true if the underlying asset pays continuous dividends at a fixed rate. In practice, continuous dividends are not a satisfying model since dividends are paid once a year or quarterly. That is why we are interested in dividends that are paid at a number of discrete points in time.

When we assume discrete dividend payments, in general, the value function of the Put option will no longer be convex in the stock price variable, even if convexity is preserved for linear dividend functions. Moreover, the optimal exercise boundary will become discontinuous at the dividend dates and before the dividend dates it may not be monotone. Integral formulas for the exercise boundary which are similar to the ones in [6] have been derived under the assumption that the boundary is Lipschitz continuous (see [8]) or locally monotonic [22]. In this paper we continue the study, undertaken in [10], of conditions under which such regularity properties of the optimal exercise boundary under discrete dividend payments can be proven.

We prove that the exercise boundary is continuous at any time which is not a dividend date and that the smooth contact property holds for the value function of the option. We considerably extend the results obtained in [10], where the continuity of the exercise boundary and the smooth contact property were only obtained in a left-hand neighborhood of the first dividend date when the corresponding dividend function was assumed to be globally concave and linear with a positive slope in a neighborhood of the origin. Under the much more restrictive assumption of global linearity of all the dividend functions, the smooth contact property and the right-continuity (resp. continuity) of the exercise boundary was proved to hold globally (resp. in a left-hand neighborhood of each dividend date). We also extend the result obtained in [10] on the decrease of the exercise boundary in a left-hand neighborhood of the first (resp. of each) dividend date when the corresponding dividend function was assumed to be positive and concave (resp. when all dividend functions were supposed to be linear): we give more general sufficient conditions on each dividend function for the exercise boundary to be either non-decreasing or non-increasing in a left-hand neighborhood of the corresponding dividend date.

In Section 1, we introduce our notations and assumptions. In the Section 2, we recall the existence results for the value function and the exercise boundary stated in [10]. Section 3 is devoted to the smooth-fit property and relies on a viscosity solution approach combined with an estimation of the derivative of the value function with respect to the time variable. In Section 4, we prove the continuity result for the exercise boundary, which is known to be upper-semicontinuous by continuity of the value function. The right-continuity is obtained by comparison with the optimal boundary of the Put option in the Black–Scholes model without dividend. The left-continuity follows from the characterization of the continuation region as the set of points where the spatial derivative of the value function is greater than -1. In Section 5, we are interested in the local behavior of the exercise boundary in a neighborhood of the dividend date. To be able to analyze this behavior, we have to assume that the stock level at which the dividend function becomes positive lies in the post-dividend exercise region. When the dividend function has a positive slope at this point, we obtain a first order expansion for the exercise boundary at the dividend date. We also provide sufficient conditions for the exercise boundary to be locally monotonic.

1. Notations and assumptions

1.1. Notations

- $\left(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}\right)$ is a probability space with a right continuous filtration, $(B_s)_{s\geq 0}$ a $(\mathcal{F}_s)$-Brownian motion under $\mathbb{P}$, and $\mathbb{Q}$ is the probability measure defined by
$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\frac{\sigma^2}{2}t+\sigma B_t}.$$
- $\bar{S}_t^x$ is a geometric Brownian motion satisfying: $d\bar{S}_t^x = r\bar{S}_t^x dt + \sigma \bar{S}_t^x dB_t$ and $\bar{S}_0^x = x$. Its density at time t is denoted $\mathrm{p}(t,x,y) = \frac{\mathbf{1}_{\{y>0\}}}{\sigma y\sqrt{2\pi t}} \exp\left(-\frac{1}{2\sigma^2 t}\left(\ln\left(\frac{y}{x}\right) - (r - \frac{\sigma^2}{2})t\right)^2\right)$,

- $\mathcal{A}$ is the Black–Scholes operator defined for any $\mathcal{C}^2$ function f by $\mathcal{A}f(x) = -rf(x) + rxf'(x) + \frac{\sigma^2 x^2}{2} f''(x)$,
- the set of all the stopping times of $(\mathcal{F}_s)_{s\leq\theta}$ is abusively denoted by $\{\tau \in [0,\theta]\}$.

Recursive construction. Let $(\theta_d^i = t_d^i - t_d^{i+1})_{0\leq i\leq I-1}$ with the convention $t_d^0 = T$ denote the durations between the dividend dates. For non-negative values of θ and x, we define by induction

- $u_0(\theta,x) = \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)^+\right]$ the price of the American Put option in the Black–Scholes model without dividends when the time to maturity is θ and the spot level x. The corresponding exercise boundary is $c_0(\theta)$ such that $\{x : u(0,x) > (K-x)^+\} = (c_0(\theta), +\infty)$. Let $v(\theta,x)$ be the value function of the American Put option with normalized strike 1 in the Black–Scholes model without dividends and $\bar{c}(\theta)$ the associated exercise boundary. One has
$$\begin{aligned} u_0(\theta,x) &= \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)^+\right] = K \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(1 - \bar{S}_\tau^{x/K}\right)^+\right] \\ &= Kv\left(\theta, \frac{x}{K}\right) \end{aligned}$$
and consequently $c_0(\theta) = \sup\left\{x | u_0(\theta,x) = (K-x)^+\right\} = K\bar{c}(\theta)$.
- $\forall i \in \{1,\dots,I\}$,
$$u_i(\theta,x) = \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)^+ \mathbf{1}_{\{\tau<\theta\}} + e^{-r\theta} u_{i-1}(\theta_d^{i-1}, \bar{S}_\theta^x - D_i(\bar{S}_\theta^x))\mathbf{1}_{\{\tau=\theta\}}\right].$$
Note that $u_i(0,x) = u_{i-1}(\theta_d^{i-1}, x - D_i(x))$.
- Any stopping time τ such that $u_i(\theta,x) = \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)^+ \mathbf{1}_{\{\tau<\theta\}} + e^{-r\theta} u_i(0, \bar{S}_\theta^x)\mathbf{1}_{\{\tau=\theta\}}\right]$ will be abusively called *an optimal stopping time for* $u_i(\theta,x)$.

1.2. Assumptions

In all what follows, we assume that

$$(A)\ \forall i \in \{1,\dots,I\}, \begin{cases} \text{(a) } D_i \text{ is non-decreasing and non-negative,} \\ \text{(b) } \rho_i : x \mapsto x - D_i(x) \text{ is non-decreasing and non-negative.} \end{cases}$$

2. Previous results

Under (A), we can reformulate Proposition 1.5 [10] with our notations.

Proposition 2.1. *Suppose that* $t < t_d^i < t_d^{i-1} < \dots < t_d^1 < T$ *and set* $\theta = t_d^i - t$, $\theta_d^0 = T - t_d^1$, *and for* $j = 1\dots i-1$, $\theta_d^j = t_d^j - t_d^{j+1}$, *then the value at time t when the spot price of the stock is equal to x of the American Put option with strike K and maturity T is given by* $u_i(\theta,x)$.

With these notations, at time $t = t_d^i$, if the spot price of the stock is x, the price of the put option is $u_{i-1}(\theta_d^{i-1}, x)$. When $D_i(x)$ is positive, it differs from $u_i(0,x) = u_{i-1}(\theta_d^{i-1}, x - D_i(x))$. The next lemma follows from Lemma 1.3 [10].

Lemma 2.2. *For each* $\theta \geq 0$, *the mapping* $x \mapsto u_i(\theta,x)$ *is non-increasing and* $x \mapsto x + u_i(\theta,x)$ *is non-decreasing.*

Like in Lemma 1.3 [10], one easily deduces the existence of the exercise boundary.

Corollary 2.3 (*Exercise Boundary*). *For $i \in \{1, \dots, I\}$ and $\theta \geq 0$, there exists $c_i(\theta) \in [0, K)$ such that $u_i(\theta, x) > (K - x)^+ \Leftrightarrow x > c_i(\theta)$*

By Proposition 2.1, the exercise boundary of the American Put option in our model with discrete dividends is

$$t \in [0, T) \mapsto \sum_{i=0}^{I} c_i(t_d^i - t)\mathbf{1}_{\left\{t_d^{i+1} \leq t < t_d^i\right\}} \quad \text{with convention } t_d^0 = T.$$

With a slight abuse of terminology, we also call exercise boundaries the functions c_i. Notice that because the argument of c_i is the time to the dividend date t_d^i, left-continuity of the c_i implies right-continuity of the true exercise boundary and that right-continuity of the c_i implies left-continuity of the true boundary on $[0, t_d^I) \cup (t_d^I, t_d^{I-1}) \cup \dots \cup (t_d^1, T)$ with existence of left-hand limits at the dividend dates.

According to Lemma 1.4 [10], one has the following.

Proposition 2.4 (*Regularity Result*). *The value function $(\theta, x) \mapsto u_i(\theta, x)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. On the continuation region defined as $\{(\theta, x) | \theta > 0,\ x > c_i(\theta)\}$, this function is $\mathcal{C}^{1,2}$ and satisfies:*

$$-\partial_\theta u_i(\theta, x) - ru_i(\theta, x) + rx\partial_x u_i(\theta, x) + \frac{\sigma^2}{2}x^2\partial_{xx}u_i(\theta, x) = 0.$$

Moreover, the left-hand derivative $\partial_{xx-}u_i(\theta, x)$ of $\partial_x u_i(\theta, \bullet)$ is well defined and equal to 0 in the exercise region $\{(\theta, x) | \theta > 0,\ 0 \leq x \leq c_i(\theta)\}$.

The upper-semi continuity of $c_i(\bullet)$ is a consequence of the continuity of u_i.

Corollary 2.5. *For any $\theta \geq 0$, $\limsup_{\theta' \to \theta} c_i(\theta') \leq c_i(\theta)$.*

Remark 2.6. Since the dividend function D_i is non-negative, $u_i(\theta, x) \geq u_{i-1}(\theta + \theta_d^{i-1}, x)$ and therefore $u_i(\theta, x) \geq u_0\left(\theta + \sum_{j=1}^{i} \theta_d^{i-1}, x\right)$. We deduce that $c_i(\theta) \leq K\bar{c}\left(\theta + \sum_{j=1}^{i} \theta_d^{i-1}\right)$. In particular, if $r = 0$, $\bar{c}(t) = 0$ for $t > 0$, so that $c_i \equiv 0$ for $i \in \{1, \dots, I\}$.

3. Smooth-fit property

In this section, we are going to prove the smooth-fit property. See [17, p. 149] for a discussion of this property in optimal stopping problems and [9, Prop. 2.8], [17, pp. 375–395] or [12, pp. 73–79] for the specific case of the American Put option in the Black–Scholes model.

Proposition 3.1 (*Smooth-fit*). *For all $\theta > 0$, $u_i(\theta, \bullet)$ is $\mathcal{C}^1$.*

The proof is based on the viscosity super-solution property of u_i and estimations of the time derivative of this function stated in the next two lemmas.

Lemma 3.2. *$(\theta, x) \mapsto u(\theta, x)$ is a viscosity supersolution of* $\min(\partial_\theta u_i(\theta, x) - \mathcal{A}u_i(\theta, \bullet)(x), u_i(\theta, x) - (K - x)^+) = 0$ *with $u_i(0, x) = u_{i-1}(\theta_d^{i-1}, \rho_i(x))$.*

Proof. It comes from the definition of u_i that $u_i(\theta, x) \geq (K-x)^+$.

Let $\phi(t,x)$ be a test function such that $0 = (u_i - \phi)(\theta, x) = \min_V (u_i - \phi)$ where $V = (\theta - \eta, \theta] \times (x-\eta, x+\eta)$ for a certain $\eta > 0$. Let τ be the first exit time of $\bar{S}^x$ outside the ball centered at x with radius η and let $0 < \epsilon < \eta$. Because of the minimum property of (θ, x), one has

$$\mathbb{E}\left[e^{-r(\tau\wedge\epsilon)}(u_i(\theta - (\tau\wedge\epsilon), \bar{S}^x_{\tau\wedge\epsilon}) - \phi(\theta - (\tau\wedge\epsilon), \bar{S}^x_{\tau\wedge\epsilon}))\right] \geq u_i(\theta, x) - \phi(\theta, x).$$

Applying the Itô formula to $e^{-rt}\phi(\theta - t, \bar{S}^x_t)$ between $t=0$ and $\tau\wedge\epsilon$, we deduce that

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\tau\wedge\epsilon} e^{-rt}(\partial_\theta\phi(\theta - t, \bar{S}^x_t) - \mathcal{A}\phi(\theta - t, \bullet)(\bar{S}^x_t))dt\right]\\ &\quad\geq \mathbb{E}\left[\left(u_i(\theta, x) - e^{-r(\tau\wedge\epsilon)}u_i(\theta - (\tau\wedge\epsilon), \bar{S}^x_{\tau\wedge\epsilon})\right)\right].\end{aligned}$$

Since, by the dynamic programming principle, for any stopping time $\eta \leq \theta$, one has $u_i(\theta, x) \geq \mathbb{E}\left[e^{-r\eta}u_i(\theta - \eta, \bar{S}^x_\eta)\right]$, the right-hand-side is non-negative. We deduce that

$$\mathbb{E}\left[\frac{1}{\epsilon}\int_0^{\tau\wedge\epsilon} e^{-rt}(\partial_\theta\phi(\theta - t, \bar{S}^x_t) - \mathcal{A}\phi(\theta - t, \bullet)(\bar{S}^x_t))dt\right] \geq 0.$$

By sending ϵ to zero, we obtain the supersolution inequality from Lebesgue's theorem:

$$\partial_\theta\phi(\theta, x) - \mathcal{A}\phi(\theta, \bullet)(x) \geq 0. \quad \square$$

Lemma 3.3. *For any $i \geq 0$, $\theta > 0$ and $x \geq 0$ one has*

$$\limsup_{\theta'\to\theta}\left|\frac{u_i(\theta', x) - u_i(\theta, x)}{\theta' - \theta}\right| \leq r(K+x) + x\left(r\left(2\mathcal{N}\left(\frac{2r}{\sigma}\sqrt{\theta}\right) - 1\right) + \sigma\frac{e^{-2\frac{r^2}{\sigma^2}\theta}}{\sqrt{2\pi\theta}}\right),$$

$$|\partial_{xx-}u_i(\theta, x)| \leq \mathbf{1}_{\{x\geq c_i(\theta)\}}\frac{2}{\sigma^2 c_i^2(\theta)}\left(2rK + \left(3r + \frac{\sigma}{\sqrt{2\pi\theta}}\right)c_i(\theta)\right).$$

Moreover $\partial_x u_i(\theta, x)$ admits a right-hand limit at $c_i(\theta)$ denoted by $\partial_x u_i(\theta, c_i(\theta)^+)$ and $\partial_x u_i(\theta, c_i(\theta)^+) \in [-1, 0]$.

The proof of these estimates, which relies on the scaling property of the Brownian motion and Lemma 2.2, is postponed in the Appendix. We are now able to prove Proposition 3.1.

Proof. Let $c = c_i(\theta)$. By Lemma 3.3, the limit $\partial_x u_i(\theta, c+) = \lim_{y\downarrow c}\partial_x u_i(\theta, y)$ exists.

We adapt a viscosity solution argument given in [18]: supposing that $\partial_x u(\theta, c+) > -1$, we are going to obtain a contradiction. For $\epsilon > 0$, let $\phi_\epsilon(x) = (K-c)^+ + \alpha(x-c) + \frac{1}{2\epsilon}(x-c)^2$ where $-1 = \partial_x u_i(\theta, c-) < \alpha < \partial_x u_i(\theta, c+)$. Since $c < K$, there exists an open interval $(x_\epsilon, y_\epsilon) \subset [0, K]$ containing c such that $\min_{x\in(x_\epsilon, y_\epsilon)}(u_i(\theta, x) - \phi_\epsilon(x)) = u_i(\theta, c) - \phi_\epsilon(c) = 0$.

We set

$$\beta = 2\left(3r + \frac{\sigma}{\sqrt{\pi\theta}}\right)K \quad \text{and} \quad \phi(\theta - t, x) = \phi_\epsilon(x) - \beta t.$$

By Lemma 3.3, for any $(t, x) \in [0, \frac{\theta}{2}] \times [0, K]$, one has $u_i(\theta - t, x) - u_i(\theta, x) \geq -\frac{\beta}{2}t$. Therefore $0 = (u_i - \phi)(\theta, c) = \min_{(t,x)\in(\frac{\theta}{2}, \theta]\times(x_\epsilon, y_\epsilon)}(u_i - \phi)(t, x)$. By the supersolution property of u_i

stated in Lemma 3.2, we deduce that

$$0 \leq \partial_\theta \phi(\theta, c) - \mathcal{A}\phi(\theta, \bullet)(c) = \beta + r(K - c) - rc\alpha - \frac{\sigma^2 c^2}{2\epsilon}.$$

By sending ϵ to zero, we get the desired contradiction. □

4. Continuity of the exercise boundary

Proposition 4.1. *Under (A), for any $i \in \{0, \ldots, I\}$, the function $\theta \mapsto c_i(\theta)$ is continuous on $[0, +\infty)$.*

The right continuity will be proved in Section 4.1 whereas the left continuity will be proved in Section 4.2.

Remark 4.2. In particular, we deduce from this result the behavior of the exercise boundary at the dividend time.

Since $c_i(0) = \sup\left\{x \geq 0 | u_{i-1}(\theta_d^{i-1}, x - D_i(x)) = K - x\right\}$ and for $y \in [0, c_{i-1}(\theta_d^{i-1}))$, $u_{i-1}(\theta_d^{i-1}, y) = K - y$, one has $c_i(0) = c_{i-1}(\theta_d^{i-1}) \wedge \inf\{x \geq 0 | D_i(x) > 0\}$ and

Corollary 4.3. *Under (A), for any $i \in \{1, \ldots, I\}$, $\lim_{t \to 0_+} c_i(t) = c_{i-1}(\theta_d^{i-1}) \wedge \inf\{x \geq 0 | D_i(x) > 0\}$.*

As $c_i(0) = 0$ when $\forall x > 0$, $D_i(x) > 0$, this result generalizes Lemma 2.1 [10].

4.1. Right continuity

The right continuity of the exercise boundary is based on a comparison result with the exercise boundary $\bar{c}$ of the classical American Put option with strike 1 in the Black–Scholes model without dividends.

Lemma 4.4. *For $\theta \geq 0$ and $t \geq 0$, one has $c_i(\theta + t) \geq \left(K\left(1 - e^{-rt}\right) + c_i(\theta)e^{-rt}\right)\bar{c}(t)$.*

Proof. Let $\tau = \tilde{\tau} \wedge t$ where $\tilde{\tau}$ is an optimal stopping time for $u_i(\theta + t, x)$. By the dynamic programming principle, one has

$$u_i(\theta + t, x) = \mathbb{E}\left[e^{-r\tau}(K - \bar{S}_\tau^x)^+ \mathbf{1}_{\{\tau < t\}} + \mathbf{1}_{\{\tau = t\}} e^{-rt} u_i(\theta, \bar{S}_t^x)\right].$$

Since $x \mapsto u_i(\theta, x)$ is non-increasing and using the fact for any $0 \leq \alpha \leq K$, $(K - x)^+ \leq (K - (\alpha \wedge x))^+ = (K - \alpha) + (\alpha - x)^+$, one deduces

$$\begin{aligned}
u_i(\theta + t, x) &\leq \mathbb{E}\left[e^{-r\tau}(K - \bar{S}_\tau^x)^+ \mathbf{1}_{\{\tau < t\}} + \mathbf{1}_{\{\tau = t\}} e^{-rt}\left(K - c_i(\theta) \wedge \bar{S}_t^x\right)^+\right] \\
&\leq \mathbb{E}\Bigg[e^{-r\tau}\left(K - \left\{c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right)\right\} \wedge \bar{S}_\tau^x\right)^+ \mathbf{1}_{\{\tau < t\}} \\
&\qquad + \mathbf{1}_{\{\tau = t\}} e^{-rt}\left(K - c_i(\theta) \wedge \bar{S}_t^x\right)^+\Bigg] \\
&\leq \mathbb{E}\left[e^{-r\tau}\left(K - \left\{c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right)\right\} \wedge \bar{S}_\tau^x\right)^+\right] \\
&\leq \mathbb{E}\left[e^{-r\tau}\left(K - \left\{c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right)\right\}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left[e^{-r\tau}\left(c_i(\theta)+(K-c_i(\theta))\left(1-e^{-r(t-\tau)}\right)-\bar{S}^x_\tau\right)^+\right]\\
&\leq (K-c_i(\theta))e^{-rt}+\mathbb{E}\left[e^{-r\tau}\left(K\left(1-e^{-rt}\right)+c_i(\theta)e^{-rt}-\bar{S}^x_\tau\right)^+\right]
\end{aligned}
$$

where we used $(K-c_i(\theta))(1-e^{-r(t-\tau)}) \leq (K-c_i(\theta))(1-e^{-rt})$ for the last inequality.

Since τ is a stopping-time not greater then t, for $x \leq \left(K\left(1-e^{-rt}\right)+c_i(\theta)e^{-rt}\right)\bar{c}(t)$, the second term of the right-hand side is not greater than $(K\left(1-e^{-rt}\right)+c_i(\theta)e^{-rt}-x)$. Therefore, one has $u_i(\theta+t,x) \leq (K-x)^+$ and $c_i(\theta+t) \geq x$. □

Corollary 4.5. *The function $\theta \mapsto c_i(\theta)$ is right continuous.*

Proof. Because $\lim_{t\to 0}\bar{c}(t) = 1$ (cf [12] pp. 71–80), Lemma 4.4 implies that $\liminf_{\theta'\downarrow\theta} c_i(\theta') \geq c_i(\theta)$. We conclude with the upper-semicontinuity property stated in Corollary 2.5. □

We recall (cf [12]) that $\bar{c}(\infty) \stackrel{\text{def}}{=} \lim_{\theta\to+\infty}\bar{c}(\theta)$ exists and is equal to $\frac{2r}{2r+\sigma^2}$.

Corollary 4.6. *One has $\lim_{\theta\to+\infty} c_i(\theta) = K\bar{c}(\infty)$. Moreover, when $r > 0$, $\forall\theta > 0$, $c_i(\theta) > 0$.*

Proof. If $r = 0$ then by Remark 2.6 the statement clearly holds.

Let us now assume that $r > 0$. Since $u_i(t,x) \geq u_0(t,x)$, we have $c_i(t) \leq K\bar{c}(t)$. Writing Lemma 4.4 for $\theta = 0$, we deduce that

$$
\forall t \geq 0, \quad -(K-c_i(0))e^{-rt}\bar{c}(t) \leq c_i(t) - K\bar{c}(t) \leq 0.
$$

We obtain the first statement by taking the limit $t \to \infty$ in this inequality.

For $\theta = 0$, Lemma 4.4 also implies $c_i(t) \geq K(1-e^{-rt})\bar{c}(t)$. Since $\bar{c}$ is non-increasing with positive limit at infinity, we deduce that $c_i(t) > 0$ as soon as $t > 0$. □

4.2. Left continuity

The left continuity is based on the characterization of the continuation region in terms of the spatial derivative of u_i stated in the next proposition.

Proposition 4.7. *Under (A), the property*

(P_i): For any $\theta > 0$ and $x \geq 0$ one has $x > c_i(\theta) \Longleftrightarrow 1 + \partial_x u_i(\theta,x) > 0$

holds for any $i \in \{0,\ldots,I\}$.

The proof of Proposition 4.7 will be done by induction on i. The main tools to deduce the induction hypothesis at rank i from the one at rank $i-1$ are in the following lemmas, the proofs of which are postponed to the Appendix.

Lemma 4.8. *Let $\theta > 0$, $x > c_i(\theta)$ and τ denote the smallest optimal stopping time for $u_i(\theta,x)$. Then $y \mapsto \mathbb{P}\left(\tau = \theta | \bar{S}^x_\theta = y\right)$ is non-decreasing and is positive on $(K,+\infty)$.*

The function $u_i(0,x)$ being Lipschitz continuous by Lemma 2.2, it is absolutely continuous and therefore dx a.e. differentiable. We denote by $\partial_x u_i(0,x)$ its a.e. derivative. For $\theta > 0$ and $x > 0$, since $\bar{S}^x_\theta$ admits a density with respect to the Lebesgue measure under $\mathbb{P}$, the random variable $\partial_x u_i(0,\bar{S}^x_\theta)$ is a.s defined under $\mathbb{P}$ and therefore under $\mathbb{Q}$.

Lemma 4.9. *Let $\theta > 0$, $x \geq 0$ and τ be an optimal stopping time for $u_i(\theta, x)$. Then one has*

$$1 + \partial_x u_i(\theta, x) \geq \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau=\theta\}}\left(1 + \partial_x u_i(0, \bar{S}^x_\theta)\right)\right].$$

Moreover, $\overline{\tau} \overset{\text{def}}{=} \lim_{\epsilon \to 0_+} \inf\left\{t \geq 0 | \bar{S}^{x+\epsilon}_t \leq c_i(\theta - t)\right\}$ *is an optimal stopping time and satisfies*

$$1 + \partial_x u_i(\theta, x) = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\overline{\tau}=\theta\}}\left(1 + \partial_x u_i(0, \bar{S}^x_\theta)\right)\right].$$

We are now proving Proposition 4.7.

Proof. First, for $i = 0$, due to [12], $x \mapsto u_i(\theta, x)$ is convex and so (P_0) is true.

Let us suppose that (P_{i-1}) holds for $i \in \{1, \ldots, I-1\}$.

By (A), $\kappa_i \overset{\text{def}}{=} \sup\left\{x \geq 0 | x - D_i(x) \leq c_{i-1}(\theta^{i-1}_d)\right\}$ is such that

$$\forall x \geq 0, \quad x - D_i(x) \leq c_{i-1}(\theta^{i-1}_d) \Leftrightarrow x \leq \kappa_i.$$

Moreover, D_i is differentiable dx a.e. and equal to the integral of its a.e. derivative which takes its values in $[0, 1]$. We denote this a.e. derivative by D'_i. Since $u_i(0, x) = u_{i-1}(\theta^{i-1}_d, x - D_i(x))$ where $u_{i-1}(\theta^{i-1}_d, x)$ is C^1 by Proposition 3.1, one easily checks that

$$dx \text{ a.e.,}\ \partial_x u_i(0, x) = (1 - D'_i(x))\partial_y u_{i-1}(\theta^{i-1}_d, y)|_{y=x-D_i(x)} \tag{1}$$

where the second term of the right-hand-side belongs to $[-1, 0]$ by Lemma 2.2. There are two possibilities:

- either $\kappa_i < \infty$ and then for $x > \kappa_i$, $1 + \partial_y u_{i-1}(\theta^{i-1}_d, y)|_{y=x-D_i(x)} > 0$ by (P_{i-1}) so that $1 + \partial_x u_i(0, x) > 0$ a.e. by Eq. (1),
- or $\kappa_i = +\infty$ and then $D_i(x) = \int_0^x D'_i(y)dy \sim x$ as $x \to \infty$. Therefore there exists a Borel set $\mathbf{C} \subset (K, +\infty)$ with infinite Lebesgue measure, on which D'_i takes values in $\left[\frac{1}{2}, 1\right]$. By Eq. (1), for almost every $x \in \mathbf{C}$, $1 + \partial_x u_i(0, x) \geq \frac{1}{2}$.

So there exists a Borel set $A \subset (K, +\infty)$ which is non neglectable for the Lebesgue measure and such that for every $x \in A$, $1 + \partial_x u_i(0, x) > 0$.

Using the first statement of Lemma 4.9 then $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_\theta} = \frac{e^{-r\theta}\bar{S}^x_\theta}{x}$, one obtains

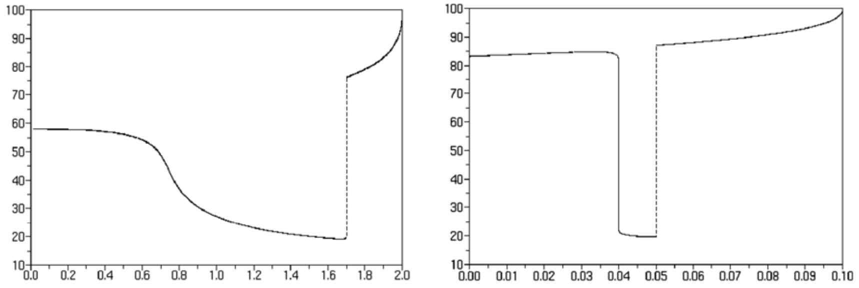
$$\begin{aligned}
1 + \partial_x u_i(\theta, x) &\geq \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau=\theta\}}\left(1 + \partial_x u_i(0, \bar{S}^x_\theta)\right)\right] \\
&= e^{-r\theta}\int_0^{+\infty} \frac{y}{x}(1 + \partial_x u_i(0, y))\,\mathbb{P}\left(\tau = \theta | \bar{S}^x_\theta = y\right)\mathrm{p}(\theta, x, y)dy \\
&\geq e^{-r\theta}\int_A \frac{y}{x}(1 + \partial_x u_i(0, y))\,\mathbb{P}\left(\tau = \theta | \bar{S}^x_\theta = y\right)\mathrm{p}(\theta, x, y)dy.
\end{aligned}$$

By Lemma 4.8, the last quantity is positive and the assertion is proved. □

Proposition 4.10. *$\theta \mapsto c_i(\theta)$ is left continuous.*

Proof. When $r = 0$, by Remark 2.6, the statement holds. Let us assume that $r > 0$. By Corollary 2.5, we just need to prove that there does not exist $\theta > 0$ such that $\liminf_{t \to 0_+} c_i(\theta - t) < c_i(\theta)$.

Let us suppose that there exists such a $\theta > 0$ and obtain a contradiction. Let $c_- \overset{\text{def}}{=} \liminf_{t \to 0_+} c_i(\theta - t)$ and $(t_n)_n$ be a decreasing sequence in $(0, \theta)$ tending to zero and such



(a) Maturity is 2 with one dividend time at 1.7; $D_1(x) = \frac{1}{5}\left((x-20)^+ - (x-30)^+\right)$.

(b) Maturity is 0.1 with one dividend time at 0.05; $D_1(x) = \min\left(\frac{9}{8}, \frac{2}{9}\left((x-20)^+\right)^2\right)$.

Fig. 2. Exercise boundaries of an American Put option with different maturities for different dividend functions. Strike is 100, diffusion parameters are $r = 0.04$ and $\sigma = 0.3$.

that $c_i(\theta - t_n)$ tend to c_-. Then, by Lemma 4.4 written with $(s - t_n, \theta - s)$ replacing (t, θ), we obtain that for $s \in (t_n, \theta)$, $c_i(\theta - s) \le c_i(\theta - t_n)\frac{e^{r(s-t_n)}}{\bar{c}(s-t_n)}$. So $\lim_{t\to 0_+} c_i(\theta - t) = c_-$. Then there exists $\eta \in (0, c_i(\theta))$, $\delta_0 \in (0, \theta/2)$, such that $\forall t \in (0, 2\delta_0)$ $c_i(\theta - t) < c_i(\theta) - \eta$. Let $x < y$ be such that $c_i(\theta) - \eta < x < y \le c_i(\theta)$. One has

$$y - x + u_i(\theta, y) - u_i(\theta, x) = 0. \tag{2}$$

Let us define $\tau = \inf\left\{t \ge 0 | t + \left|\bar{S}^1_t - 1\right| \ge \delta_0 \wedge \frac{x - c_i(\theta) + \eta}{x}\right\}$. For $\theta' \in (\theta, \theta - \delta_0)$ and $z \ge x$, one has $\forall t \in [0, \tau]$, $\bar{S}^z_t \ge \bar{S}^x_t \ge c_i(\theta) - \eta > c_i(\theta' - t)$ and by Proposition 2.4, $u_i(\theta', z) = \mathbb{E}\left[e^{-r\tau} u_i(\theta' - \tau, \bar{S}^z_\tau)\right]$. Since u_i is continuous and bounded by K, letting θ' tend to θ, we get by dominated convergence $u_i(\theta, z) = \mathbb{E}\left[e^{-r\tau} u_i(\theta - \tau, \bar{S}^z_\tau)\right]$. We deduce

$$\begin{aligned} y - x + u_i(\theta, y) - u_i(\theta, x) &= \mathbb{E}\left[e^{-r\tau}\left(\bar{S}^y_\tau - \bar{S}^x_\tau + u_i(\theta - \tau, \bar{S}^y_\tau) - u_i(\theta - \tau, \bar{S}^x_\tau)\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\int_x^y \left(1 + \partial_x u_i(\theta - \tau, \bar{S}^z_\tau)\right) dz\right]. \end{aligned}$$

But since $\mathbb{Q}\left(\tau > 0 \text{ and } \forall z \ge x,\ \bar{S}^z_\tau > c_i(\theta - \tau)\right) = 1$, the right-hand side is positive by Proposition 4.7, which contradicts Eq. (2). □

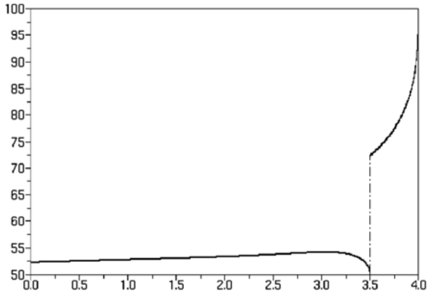
In Fig. 2, we represent two different exercise boundaries computed through a binomial tree method following [21]. In both cases, $c_1(0) = \kappa_1 = 20$. In case (a), the boundary appears to be smooth whereas in case (b), it seems to be merely continuous (at time 0.04, even continuity is not so clear from the figure).

5. Local behavior of the exercise boundary near the dividend dates

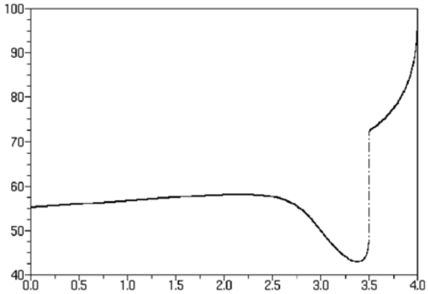
In this section, we are going to show how the behavior of the exercise boundary is driven by the shape of the function $u_i(0, .)$ when $i \in \{1, \dots, I\}$.

We recall that $c_i(0) = \min\left(c_{i-1}(\theta_d^{i-1}), \inf\{x \ge 0 | D_i(x) > 0\}\right)$. Applying Lemma 4.4 for $\theta = 0$ and $t = \theta_d^{i-1}$, one obtains

$$c_{i-1}(\theta_d^{i-1}) \ge \left(K(1 - e^{-r\theta_d^{i-1}}) + e^{-r\theta_d^{i-1}} c_{i-1}(0)\right)\bar{c}(\theta_d^{i-1}) \ge \frac{2rK}{2r + \sigma^2}(1 - e^{-r\theta_d^{i-1}}). \tag{3}$$



(a) $D_1(x) = \min\left(\frac{1125}{32}, \frac{8}{1125}\left((x-50)^+\right)^2\right)$.



(b) $D_1(x) = 0.05(x-50)^+$.

Fig. 3. Exercise boundaries of an American Put option of maturity 4 with one dividend time at 3.5 for different dividend functions. Strike is 100, diffusion parameters are $r = 0.04$ and $\sigma = 0.3$.

We are able to precise the local behavior of the exercise boundary near the dividend dates only when $c_i(0) < c_{i-1}(\theta_d^{i-1})$ which is satisfied as soon as $\inf\{x \geq 0 | D_i(x) > 0\} < \frac{2rK}{2r+\sigma^2}(1 - e^{-r\theta_d^{i-1}})$. In Fig. 3 are represented two different exercise boundaries computed through a binomial tree method following [21]. Notice that in each case, a dividend is paid if the stock price is over 50. On the left (resp. right) one, $c_1(.)$ seems to be locally increasing (resp. decreasing) on $[0, \epsilon)$ for ϵ small enough. In Propositions 5.3 and 5.6, we give sufficient conditions on the dividend functions for these local monotonicity properties to hold.

5.1. Equivalent of the exercise boundary for dividend functions with positive slope at $c_i(0)_+$

Proposition 5.1. *If $c_i(0) > 0$ and $\liminf_{x \to c_i(0)_+} \frac{D_i(x)}{x - c_i(0)} > 0$, then $c_i(\theta) - c_i(0) \sim_{\theta \to 0_+} -\sigma c_i(0)\sqrt{\theta\,|\ln\theta|}$.*

By Remark 2.6 and Eq. (3), a necessary and sufficient condition for the positivity of $c_i(0)$ is positivity of both r and $\inf\{x \geq 0 | D_i(x) > 0\}$. Notice that the second hypothesis implies that $c_i(0) = \inf\{x \geq 0 | D_i(x) > 0\}$ and therefore that $\inf\{x \geq 0 | D_i(x) > 0\} \leq c_{i-1}(\theta_d^{i-1})$ with possible equality. In order to prove Proposition 5.1, we need the following lemma, the proof of which is postponed in the Appendix.

Lemma 5.2. *Suppose that $c_i(0) > 0$ and that there exists $\alpha > 0$, $\beta \in [1, 2)$ and an open set $V \subset \mathbb{R}_+^\star$ containing $c_i(0)$ such that*

$$\forall x \in V, \quad u_i(0, x) - (K - x)^+ \geq \alpha \left|(x - c_i(0))^+\right|^\beta. \tag{4}$$

Then $\forall \delta > 1,\ \exists \Theta_\delta > 0,\ \forall \theta \in [0, \Theta_\delta]$,

$$c_i(\theta) \leq c_i(0) \exp\left\{-\sigma\sqrt{\theta\left((2-\beta)\,|\ln\theta| - (\beta+\delta)\ln|\ln\theta|\right)}\right\}.$$

In particular, when $D_i(x) = \alpha(x - \beta)^+ \wedge \gamma$ with $\alpha \in (0, 1)$, $\beta \in (0, c_{i-1}(\theta_d^{i-1})]$ and $\gamma > 0$, in a neighborhood of 0, the exercise boundary c_i is under a decreasing function coinciding with $c_i(0)$ at 0.

We are now able to prove Proposition 5.1.

Proof. Since $c_i(0) \leq c_{i-1}(\theta_d^{i-1}) < K$ and for $x \in [0, K]$, $u_{i-1}(0, x) \geq K - x + D_i(x)$, the positivity of $\liminf_{x \to c_i(0)_+} \frac{D_i(x)}{x - c_i(0)}$ implies that the second hypothesis of Lemma 5.2 is satisfied with $\beta = 1$. Hence, for θ small enough, $c_i(\theta) \le c_i(0)e^{-\sigma\sqrt{\theta(|\ln\theta|-3\ln|\ln\theta|)}}$. By Lemma 4.4, we know that $c_i(\theta) \ge c_i(0)\bar{c}(\theta) + (1-e^{-r\theta})(K - c_i(0))\,\bar{c}(\theta)$, where, according to [13], $\bar{c}(\theta) - 1 \sim_{\theta\downarrow 0} -\sigma\sqrt{\theta\,|\ln\theta|}$. Since $\sqrt{\theta\,(|\ln\theta| - 3\ln|\ln\theta|)} \sim_{\theta\downarrow 0} \sqrt{\theta\,|\ln\theta|}$, we easily conclude. $\square$

5.2. *Monotonicity of the value function*

The monotonicity of the value function around the i-th dividend time is closely related to the sign, on a right-hand neighborhood of $c_i(0)$, of the Black–Scholes operator applied to $u_i(0,.) = u_{i-1}(\theta_d^{i-1}, \rho_i(.))$ where $\rho_i(x) = x - D_i(x)$. In the previous sections, the derivative of D_i was thought in the sense of distributions. From now on, we assume that D_i is the difference of two convex functions in order to apply the Itô–Tanaka formula. So the derivative of D_i (resp. ρ_i) is considered as the left-hand derivative.

5.2.1. *Exercise boundary locally non-decreasing*

To obtain this property, we need negativity of the Black–Scholes operator applied to $u_i(0,.)$ in a right-hand neighborhood of $c_i(0)$.

Proposition 5.3. *Assume that* $\inf\{x \ge 0 | D_i(x) > 0\} < c_{i-1}(\theta_d^{i-1})$, *that* D_i *is the difference of two convex functions, and that the positive part of the Jordan–Hahn decomposition of the measure* D_i'' *is absolutely continuous with respect to the Lebesgue measure. Assume moreover that, if* g_i *denotes the density of the absolutely continuous part of* D_i'', *there exists* $\varepsilon \in (0, c_{i-1}(\theta_d^{i-1}) - c_i(0))$ *and* $C_1 \in [0,+\infty)$ *such that*

$$\forall x \le c_i(0)+\varepsilon, \quad -rD_i(x) + rxD_i'(x) + \frac{\sigma^2x^2}{2}g_i(x) \le rK - \varepsilon$$
$$\forall x > c_i(0)+\varepsilon, \quad g_i(x) \le C_1x^{C_1}.$$

Then there exists a neighborhood of $(0, c_i(0))$ *in* $\mathbb{R}_+ \times \mathbb{R}_+$ *such that* u_i *is non-increasing w.r.t* θ *in this neighborhood. Moreover, the exercise boundary* c_i *is non-decreasing in a neighborhood of* 0.

According to Eq. (3), when $\inf\{x \ge 0 | D_i(x) > 0\} < \frac{2rK}{2r+\sigma^2}(1 - e^{-r\theta_d^{i-1}})$, then $\inf\{x \ge 0 | D_i(x) > 0\} < c_{i-1}(\theta_d^{i-1})$.

Remark 5.4. This result is a generalization of Proposition 2.2 in [10] which states the same local monotonicity property of the value function at the first dividend date when $c_1(0) = 0$ and D_1 is a non-zero concave function satisfying assumption (A). Indeed concavity implies that $g_1(x) \le 0$ and $D_1(x) - rxD_1'(x) \ge D_1(0)$ where $D_1(0) = 0$ by (A). When $r > 0$ and $c_i(0) = 0$, generalizing the proofs of Lemma 2.1 and Corollary 2.3 [10], one may check that $c_i(\theta) \le rK\theta \limsup_{x\to 0^+} \frac{x}{D_i(x)} + o(\theta)$ as $\theta \to 0$ and that, under the assumptions of Proposition 5.3, if $\frac{x}{D_i(x)}$ admits a finite right-hand limit at $x = 0$, $c_i(\theta) \sim_{\theta\to 0^+} rK\theta \lim_{x\to 0^+} \frac{x}{D_i(x)}$.

When $r > 0$, for $\beta \in (0, c_{i-1}(\theta_d^{i-1}))$, $\eta \in (0, r)$ and $\alpha \in (0, \frac{\sigma^2\beta^2}{4(r-\eta)K}]$, the function $D_i(x) = \min\left(\alpha, \frac{(r-\eta)K}{\sigma^2\beta^2}\left((x-\beta)^+\right)^2\right)$ satisfies (A) and the assumptions of Proposition 5.3.

To prove the proposition, we need the following lemma, the proof of which is postponed in the Appendix.

Lemma 5.5. *For $t_1 \geq 0$, let $\tau_{t_1} = \inf\left\{w \geq 0 | \bar{S}^x_w \geq c_i(t_1 - w)\mathbf{1}_{\{w<t_1\}} + c_i(0)\mathbf{1}_{\{w \geq t_1\}}\right\}$ with the convention* $\inf \emptyset = +\infty$.

$$\forall p \geq 0,\ \forall \alpha > 0,\ \exists \eta > 0,$$
$$\lim_{v \to 0+} \sup_{t_1 \leq \eta} \sup_{x \leq c_i(0)+\alpha} \frac{\mathbb{E}\left[\left(1 + (\bar{S}^x_v)^p\right)\mathbf{1}_{\{\tau_{t_1} \geq v, \bar{S}^x_v \geq c_i(0)+2\alpha\}}\right]}{\mathbb{P}(\tau_{t_1} \geq v)} = 0.$$

We are now able to prove Proposition 5.3.

Proof. Let $0 \leq s < t$, $x > c_i(t)$ and τ be the smallest optimal stopping time for (t, x). Since $\tau \wedge s$ is a stopping time not greater than s, $u_i(s, x) \geq \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}^x_\tau\right)\mathbf{1}_{\{\tau<s\}} + e^{-rs}u_i(0, \bar{S}^x_s)\right]$. Using $(K - x)^+ \leq u_i(0, x)$, we deduce

$$u_i(t, x) - u_i(s, x) \leq \mathbb{E}\left[\mathbf{1}_{\{\tau \geq s\}}\left(e^{-r\tau}u_i(0, \bar{S}^x_\tau) - e^{-rs}u_i(0, \bar{S}^x_s)\right)\right].$$

By Lemma A.1, on $\tau > s$,

$$\begin{aligned} &e^{-r\tau}u_i(0, \bar{S}^x_\tau) - e^{-rs}u_i(0, \bar{S}^x_s) \\ &\quad = \int_s^\tau e^{-rv}\left\{-ru_i(0, \bar{S}^x_v) + r\bar{S}^x_v \partial_x u_{i-1}(\theta^{i-1}_d, \rho_i(\bar{S}^x_v))\rho'_i(\bar{S}^x_v)\right. \\ &\qquad \left. + \frac{\sigma^2}{2}\left(\bar{S}^x_v \rho'_i\left(\bar{S}^x_v\right)\right)^2 \partial_{xx}u_{i-1}(\theta^{i-1}_d, \rho_i(\bar{S}^x_v))\right\} dv \\ &\qquad + \frac{1}{2}\int_s^\tau \int_{\mathbb{R}} e^{-rv}\partial_x u_{i-1}(\theta^{i-1}_d, \rho_i(a))\rho''_i(da)dL^a_v(\bar{S}^x) + M_\tau - M_s \end{aligned} \tag{5}$$

where $M_t = \int_0^t \sigma e^{-rv}\bar{S}^x_v \partial_x u_{i-1}(\theta^{i-1}_d, \rho_i(\bar{S}^x_v))\rho'_i(\bar{S}^x_v)dB_v$. As $\mathbb{E}\left[\langle M\rangle_t\right] \leq \sigma^2 t x^2 e^{\sigma^2 t}$, M_t is a true martingale and

$$\mathbb{E}\left[\mathbf{1}_{\{\tau \geq s\}}(M_\tau - M_s)\right] = \mathbb{E}\left[\mathbf{1}_{\{\tau \geq s\}}(\mathbb{E}\left[M_\tau | \mathcal{F}_s\right] - M_s)\right] = 0. \tag{6}$$

The function $y \mapsto \partial_x u_{i-1}(\theta^{i-1}_d, \rho_i(y))$ belongs to $[-1, 0]$ by Lemma 2.2 and is equal to -1 on $[0, c_i(0) + \varepsilon]$ since then $\rho_i(y) \leq y \leq c_i(0) + \varepsilon < c_{i-1}(\theta^d_{i-1})$. Since for any $a \geq 0$, $t \mapsto L^a_t$ is a non-decreasing process and $\rho''_i = -D''_i$, using the growth assumption on g_i, we deduce that $\mathbb{P}$-almost surely

$$\begin{aligned} &\int_s^\tau \int_{\mathbb{R}} e^{-rv}\partial_x u_{i-1}(\theta^{i-1}_d, \rho_i(a))\rho''_i(da)dL^a_v(\bar{S}^x) \\ &\quad \leq \int_s^\tau \int_{\mathbb{R}} e^{-rv}\left(\mathbf{1}_{\{a \leq c_i(0)+\varepsilon\}}g_i(a) + \mathbf{1}_{\{a > c_i(0)+\varepsilon\}}C_1 a^{C_1}\right) da dL^a_v(\bar{S}^x). \end{aligned}$$

Using Exercise 1.15 p. 232 [19], we deduce that

$$\begin{aligned} &\int_s^\tau \int_{\mathbb{R}} e^{-rv}\partial_x u_{i-1}(\theta^{i-1}_d, \rho_i(a))\rho''_i(da)dL^a_v(\bar{S}^x) \\ &\quad \leq \int_s^\tau \sigma^2 e^{-rv}\left(\bar{S}^x_v\right)^2 (\mathbf{1}_{\{\bar{S}^x_x \leq c_i(0)+\varepsilon\}}g_i(\bar{S}^x_v) + C_1\mathbf{1}_{\{\bar{S}^x_v > c_i(0)+\varepsilon\}}(\bar{S}^x_v)^{C_1})dv. \end{aligned} \tag{7}$$

By Lemma 3.3 and since $c_i(0)+\varepsilon < c_{i-1}(\theta_d^{i-1})$, there exists a finite constant C_2 not depending on s and t such that

$$\int_s^\tau e^{-rv}\left(\bar{S}_v^x\rho_i'\left(\bar{S}_v^x\right)\right)^2\partial_{xx}u_{i-1}(\theta_d^{i-1},\rho_i(\bar{S}_v^x))dv \le C_2\int_s^\tau e^{-rv}\left(\bar{S}_v^x\right)^2\mathbf{1}_{\{\bar{S}_v^x>c_i(0)+\varepsilon\}}dv. \quad (8)$$

For $y \le c_i(0)+\varepsilon$, $u_{i-1}(\theta_d^{i-1},\rho_i(y)) = K-\rho_i(y)$ and

$$-ru_i(0,y)+ry\partial_x u_{i-1}(\theta_d^{i-1},\rho_i(y))\rho_i'(y) = -rK - rD_i(y) + ryD_i'(y)$$

where D_i is equal to 0 on $[0, c_i(0)]$. Hence the assumptions ensure that

$$\begin{aligned} &-ru_i(0,y)+ry\partial_x u_{i-1}(\theta_d^{i-1},\rho_i(y))\rho_i'(y)+\frac{\sigma^2y^2}{2}g_i(y)\\ &\le\begin{cases}-rK & \text{if } y\le c_i(0)\\ -\varepsilon & \text{if } y\in(c_i(0),c_i(0)+\varepsilon].\end{cases} \end{aligned} \quad (9)$$

When $y > c_i(0)+\varepsilon$, since $\partial_x u_{i-1}\le 0$ and $\rho_i'\ge 0$, $-ru_i(0,y)+ry\partial_x u_{i-1}(\theta_d^{i-1},\rho_i(y))\rho_i'(y)$ is non-positive.

Taking expectations in Eq. (5) and using Eqs. (6)–(9), we deduce that there exists a constant $M>0$ such that

$$\begin{aligned} u_i(t,x)-u_i(s,x) \le &\int_s^t e^{-rv}\mathbb{P}(\tau\ge v)\\ &\times\left\{-(rK\wedge\varepsilon)+M\frac{\mathbb{E}\left[\mathbf{1}_{\{\tau\ge v,\bar{S}_v^x>c_i(0)+\varepsilon\}}\left(1+\left(\bar{S}_v^x\right)^{2+C_1}\right)\right]}{\mathbb{P}(\tau\ge v)}\right\}dv. \end{aligned} \quad (10)$$

Applying Lemma 5.5 (with $p=2+C_1$, $t_1=t$ and $\alpha=\frac{\varepsilon}{2}$), we obtain that for t small enough, uniformly in $x\le c_i(0)+\frac{\varepsilon}{2}$, the right-hand-side of Eq. (10) is non-positive.

With Proposition 4.1, we deduce the existence of $\eta>0$ such that $\sup_{w\in[0,\eta]}c_i(w)\le c_i(0)+\frac{\varepsilon}{2}$ and

$$\forall 0\le s<t<\eta,\ \forall x\in\left(c_i(t),c_i(0)+\frac{\epsilon}{2}\right],\quad u_i(t,x)\le u_i(s,x).$$

This inequality is still true for $x\le c_i(t)$ since then $u_i(t,x)=(K-x)^+\le u_i(s,x)$. For $0\le s<t<\eta$, we conclude that $u_i(t,c_i(s))\le u_i(s,c_i(s)) = K-c_i(s)$, which implies that $c_i(s)\le c_i(t)$. □

5.2.2. *Exercise boundary locally non-increasing*

To obtain this property, we need positivity of the Black–Scholes operator applied to $u_i(0,.)$ in a right-hand neighborhood of $c_i(0)$.

Proposition 5.6. *Assume that* $0<\inf\{x\ge 0|D_i(x)>0\}<c_{i-1}(\theta_d^{i-1})$, *that* D_i *is the difference of two convex functions, and that the negative part of the Jordan–Hahn decomposition of the measure* D_i'' *is absolutely continuous with respect to the Lebesgue measure.*

Assume moreover that, if g_i *denotes the density of the absolutely continuous part of the measure* D_i'', *there exists* $\varepsilon\in(0,c_{i-1}(\theta_d^{i-1})-c_i(0))$ *and* $C_1\in[0,+\infty)$ *such that*

on $(c_i(0),c_i(0)+\varepsilon]$, D_i *is* C^2 *and such that*

$$-rD_i(x) + rxD_i'(x) + \frac{\sigma^2 x^2}{2} g_i(x) \geq rK + \varepsilon,$$
$$\forall x > c_i(0) + \varepsilon,\ \ g_i(x) \leq -C_1 x^{C_1}.$$

Then there exists a neighborhood of $(0, c_i(0))$ *in* $\mathbb{R}_+ \times \mathbb{R}_+$ *such that* u_i *is non-decreasing w.r.t* θ *in this neighborhood. Moreover the exercise boundary* c_i *is non-increasing in a neighborhood of* 0.

Remark 5.7. When $c_i(0) = 0$, there is no non-negative function D_i satisfying the differential inequality on some interval $(c_i(0), c_i(0)+\varepsilon)$. That is why we suppose $\inf\{x \geq 0 | D_i(x) > 0\} > 0$ in the previous proposition.

When $r > 0$, $\beta \in (0, c_{i-1}(\theta_d^{i-1}))$ and $\alpha \in (0, 1)$, the function

$$D_i(x) = \alpha(x-\beta)^+ + \left(\frac{1}{\sigma\beta}\right)^2 (r(K-\alpha\beta)+\eta)\left((x-\beta)^+\right)^2 e^{-\frac{x^2}{\eta}} \tag{11}$$

satisfies (A) and the assumptions of Proposition 5.6 when $\eta > 0$ is small enough.

Unfortunately, Proposition 5.6 does not apply to the simple dividend function $\alpha(x-\beta)^+$ without addition of the second term in the right-hand-side of (11), even if from Fig. 3(b) and the sentence following Lemma 5.2, one expects local monotonicity of the boundary.

Proof. Let $0 \leq s < t$, $x > c_i(s)$ and τ be the smallest optimal stopping time for (s, x). We set $\bar{\tau} = \tau \mathbf{1}_{\{\tau<s\}} + \mathbf{1}_{\{\tau=s\}}\left(\inf\left\{v \geq s | \bar{S}_v^x \leq c_i(0)\right\} \wedge t\right)$. We have

$$u_i(t,x) - u_i(s,x) \geq \mathbb{E}\left[\mathbf{1}_{\{\tau=s\}}\left(e^{-r\bar{\tau}} u_i(t-\bar{\tau}, \bar{S}_{\bar{\tau}}^x) - e^{-rs} u_i(0, \bar{S}_s^x)\right)\right].$$

Since on $\{\tau = s\}$, $\bar{S}_s^x \geq c_i(0)$, on $\{\tau = s, \bar{\tau} < t\}$, $\bar{S}_{\bar{\tau}}^x = c_i(0)$, and $u_i(t-\bar{\tau}, c_i(0)) \geq (K - c_i(0)) = u_i(0, c_i(0))$. We then deduce that

$$u_i(t,x) - u_i(s,x) \geq \mathbb{E}\left[\mathbf{1}_{\{\bar{\tau}\geq s\}}\left(e^{-r\bar{\tau}} u_i(0, \bar{S}_{\bar{\tau}}^x) - e^{-rs} u_i(0, \bar{S}_s^x)\right)\right].$$

Applying Lemma A.1, arguing like in the proof of Proposition 5.3 about the local martingale part and using that dv a.e. on $[s,t]$, $\bar{\tau} \geq v$ implies $\bar{S}_v^x > c_i(0)$, we get

$$\begin{aligned}
&u_i(t,x) - u_i(s,x) \\
&\geq \mathbb{E}\left[\int_s^t \mathbf{1}_{\{\bar{\tau}\geq v, \bar{S}_v^x > c_i(0)\}} e^{-rv}\left\{-ru_i(0, \bar{S}_v^x) + r\bar{S}_v^x \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x))\rho_i'(\bar{S}_v^x)\right.\right. \\
&\quad \left.\left. + \frac{\sigma^2}{2}\left(\bar{S}_v^x \rho_i'\left(\bar{S}_v^x\right)\right)^2 \partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x))\right\} dv\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\int_s^t \int_{\mathbb{R}} \mathbf{1}_{\{\bar{\tau}\geq v\}} e^{-rv} \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a))\rho_i''(da) dL_v^a(\bar{S}^x)\right].
\end{aligned}$$

Like in the proof of Proposition 5.3, one checks that

$$\forall y \in (c_i(0), c_i(0)+\varepsilon], \quad -ru_i(0,y) + ry\partial_x u_i(\theta_d^{i-1}, \rho_i(y))\rho_i'(y) + \frac{\sigma^2 y^2}{2} g_i(y) \geq \varepsilon$$

$$\forall y > c_i(0)+\varepsilon, \quad -ru_i(0,y) + ry\partial_x u_i(\theta_d^{i-1}, \rho_i(y))\rho_i'(y) \geq -r(K+y),$$

$$\int_s^t \mathbf{1}_{\{\bar{\tau}\geq v\}} e^{-rv} \left(\bar{S}_v^x \rho_i' \left(\bar{S}_v^x\right)\right)^2 \partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x)) dv$$

$$\geq -C_2 \int_s^t \mathbf{1}_{\{\bar{\tau}\geq v, \bar{S}_v^x > c_i(0)+\varepsilon\}} e^{-rv} \left(\bar{S}_v^x\right)^2 dv,$$

and that

$$\int_s^t \int_{\mathbb{R}} \mathbf{1}_{\{\bar{\tau}\geq v\}} e^{-rv} \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a)) \rho_i''(da) dL_v^a(\bar{S}^x)$$
$$\geq \int_s^t \mathbf{1}_{\{\bar{\tau}\geq v\}} e^{-rv} \sigma^2 \left(\bar{S}_v^x\right)^2 \left[g_i(\bar{S}_v^x) \mathbf{1}_{\{\bar{S}_v^x \leq c_i(0)+\varepsilon\}} - C_1 (\bar{S}_v^x)^{C_1} \mathbf{1}_{\{\bar{S}_v^x > c_i(0)+\varepsilon\}} \right] dv.$$

Gathering all the inequalities, we get that there exists a finite constant $M \geq 0$ such that

$$u_i(t,x) - u_i(s,x)$$
$$\geq \int_s^t \left\{ \mathbb{P}(\bar{\tau} \geq v) e^{-rv} \varepsilon - \mathbb{E}\left[\mathbf{1}_{\{\bar{\tau}\geq v, \bar{S}_v^x > c_i(0)+\varepsilon\}} M \left(1 + \left(\bar{S}_v^x\right)^{2+C_1}\right)\right]\right\} dv. \tag{12}$$

Applying Lemma 5.5 (with $p = 2 + C_1$, $t_1 = s$ and $\alpha = \frac{\varepsilon}{2}$), we obtain that for t small enough, uniformly for $x \leq c_i(0) + \frac{\varepsilon}{2}$, the right-hand-side of Eq. (12) is non-negative.

With Proposition 4.1, we deduce the existence of $\eta > 0$ such that $\sup_{w\in[0,\eta]} c_i(w) \leq c_i(0) + \frac{\varepsilon}{2}$ and that

$$\forall 0 \leq s < t < \eta, \forall x \in \left(c_i(s), c_i(0) + \frac{\varepsilon}{2}\right), \quad u_i(s,x) \leq u_i(t,x).$$

This inequality is still true for $x \leq c_i(s)$ since then $u_i(s,x) = (K-x)^+ \leq u_i(t,x)$.

Then, as soon as $0 \leq s < t < \eta$, $u_i(s, c_i(t)) \leq u_i(t, c_i(t)) = K - c_i(t)$ which implies that $c_i(t) \leq c_i(s)$. □

6. Conclusion and further research

The continuity of the exercise boundary as well as the smooth contact property are likely to be generalized in a model with discrete dividends where the underlying asset price has a local volatility dynamics between the dividend dates with a positive local volatility function. We plan to investigate this extension in a future work. Assuming that the underlying stock price evolves as the exponential of some Lévy process between the dividend dates provides another natural generalization of the Black–Scholes model that could be considered (see [14] for the case without discrete dividends).

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Appendix

A.1. *Proof of Lemma 3.3*

Proof. The existence of the right-hand limit at $c_i(\theta)$ for $\partial_x u_i(\theta,x)$ is an easy consequence of the second estimation. Since for $x < c_i(\theta)$, $\partial_{xx} u_i(\theta,x) = 0$ and for $x > c_i(\theta)$, by Proposition 2.4

and Lemma 2.2,

$$
\begin{aligned}
|\partial_{xx} u_i(\theta, x)| &= \left| \frac{2}{\sigma^2 x^2} \left(\partial_\theta u_i(\theta, x) + r u_i(\theta, x) - r x \partial_x u_i(\theta, x) \right) \right| \\
&\le \frac{2}{\sigma^2 x^2} |\partial_\theta u_i(\theta, x)| + \frac{2r}{\sigma^2} \left(\frac{K}{x^2} + \frac{1}{x} \right),
\end{aligned}
$$

the second estimation is easily deduced from the first one. To prove the first estimation, we set

$$
\begin{aligned}
V_i : (\gamma, \nu, x) \mapsto \sup_{\tau \in [0,1]} \mathbb{E} \Bigg[& e^{-\gamma \frac{\nu^2}{2} \tau} \left(K - x e^{\frac{\nu^2}{2}(\gamma - 1)\tau + \nu B_\tau} \right)^+ \mathbf{1}_{\{\tau < 1\}} \\
& + e^{-\gamma \frac{\nu^2}{2}} u_i(0, x e^{\frac{\nu^2}{2}(\gamma-1) + \nu B_1}) \mathbf{1}_{\{\tau = 1\}} \Bigg].
\end{aligned}
$$

Because of the scaling property of the Brownian motion, for any positive $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\theta \in \mathbb{R}^+$, $\sup_{\tau \in [0,1]} \mathbb{E}\left[f(\theta\tau, \sqrt{\theta} B_\tau) \right] = \sup_{\tau \in [0,\theta]} \mathbb{E}[f(\tau, B_\tau)]$.

We deduce that $V_i\left(\frac{2r}{\sigma^2}, \sigma\sqrt{\theta}, x\right) = u_i(\theta, x)$ and

$$
\limsup_{\theta' \to \theta} \left| \frac{u_i(\theta', x) - u_i(\theta, x)}{\theta' - \theta} \right| = \frac{\sigma}{2\sqrt{\theta}} \limsup_{\nu' \to \sigma\sqrt{\theta}} \left| \frac{V_i(\frac{2r}{\sigma^2}, \nu', x) - V_i(\frac{2r}{\sigma^2}, \sigma\sqrt{\theta}, x)}{\nu' - \sigma\sqrt{\theta}} \right|.
$$

Therefore it is enough to check that

$$
\begin{aligned}
\forall x, \nu \ge 0, \qquad & \limsup_{\nu' \to \nu} \left| \frac{V_i(\gamma, \nu', x) - V_i(\gamma, \nu, x)}{\nu' - \nu} \right| \\
& \le \nu\gamma (K + x) + x \left(\gamma\nu (2\mathcal{N}(\gamma\nu) - 1) + 2 \frac{e^{-\gamma^2 \frac{\nu^2}{2}}}{\sqrt{2\pi}} \right). \qquad (13)
\end{aligned}
$$

Setting $(\gamma, \nu) = (\frac{2r}{\sigma^2}, \sigma\sqrt{\theta})$, the optimality of $\tau = \inf\left\{ t \ge 0 | u_i(\theta - t, \bar{S}_t^x) + \bar{S}_t^x \le K \right\} \wedge \theta$ for $u_i(\theta, x)$ translates into the optimality of

$$
\tau^\star \overset{\text{def}}{=} \inf \left\{ t \ge 0 | V_i(\gamma, \nu\sqrt{1-t}, x e^{\frac{\nu^2}{2}(\gamma-1)t + \nu B_t}) + x e^{\frac{\nu^2}{2}(\gamma-1)t + \nu B_t} \le K \right\} \wedge 1
$$

for $V_i(\gamma, \nu, x)$. This implies that

$$
\begin{aligned}
V_i(\gamma, \nu, x) + x = {} & K \mathbb{E}\left[e^{-\frac{\nu^2}{2}\gamma\tau^\star} \right] \\
& + \mathbb{E}\left[\mathbf{1}_{\{\tau^\star = 1\}} e^{-\frac{\nu^2}{2}\gamma} \left(u_i(0, x e^{\frac{\nu^2}{2}(\gamma-1) + \nu B_1}) + x e^{\frac{\nu^2}{2}(\gamma-1) + \nu B_1} - K \right) \right].
\end{aligned}
$$

For any $\nu' \ge 0$, by definition of V_i,

$$
\begin{aligned}
V_i(\gamma, \nu', x) + x \ge {} & K \mathbb{E}\left[e^{-\frac{\nu'^2}{2}\gamma\tau^\star} \right] \\
& + \mathbb{E}\left[\mathbf{1}_{\{\tau^\star = 1\}} e^{-\frac{\nu'^2}{2}\gamma} \left(u_i(0, x e^{\frac{\nu'^2}{2}(\gamma-1) + \nu' B_1}) + x e^{\frac{\nu'^2}{2}(\gamma-1) + \nu' B_1} - K \right) \right].
\end{aligned}
$$

Using that $x \mapsto x + u_i(0,x)$ is 1-Lipschitz and non-decreasing by Lemma 2.2, then $u_i(0,.) \le K$ and $(1-e^x)^+ \le (-x)^+ \le |x|$, one deduces

$$\begin{aligned}
V_i(\gamma,\nu',x) - V_i(\gamma,\nu,x) \ge\ & K\mathbb{E}\left[\left\{e^{-\frac{\nu'^2}{2}\gamma\tau^\star} - e^{-\frac{\nu^2}{2}\gamma\tau^\star}\right\}\mathbf{1}_{\{\tau^\star<1\}}\right] \\
&+ \left[e^{-\frac{\nu'^2}{2}\gamma} - e^{-\frac{\nu^2}{2}\gamma}\right]\mathbb{E}\left[\mathbf{1}_{\{\tau^\star=1\}}\left(u_i(0, xe^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}) + xe^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}\right)\right] \\
&- e^{-\frac{\nu'^2}{2}\gamma}\mathbb{E}\left[\mathbf{1}_{\{\tau^\star=1\}}xe^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}\left(1 - e^{(\nu'-\nu)\left((\gamma-1)\frac{\nu+\nu'}{2}+B_1\right)}\right)^+\right]. \\
\ge\ & -K(e^{-\frac{\nu^2}{2}\gamma} - e^{-\frac{\nu'^2}{2}\gamma})^+(\mathbb{P}(\tau^\star<1) + \mathbb{P}(\tau^\star=1)) \\
&- x\left(1 - e^{\frac{\nu^2-\nu'^2}{2}\gamma}\right)^+\mathbb{E}\left[\mathbf{1}_{\{\tau^\star=1\}}e^{-\frac{\nu^2}{2}+\nu B_1}\right] \\
&- e^{\frac{\nu^2-\nu'^2}{2}\gamma}\left|\nu'-\nu\right|\mathbb{E}\left[\mathbf{1}_{\{\tau^\star=1\}}xe^{-\frac{\nu^2}{2}+\nu B_1}\left|(\gamma-1)\frac{\nu+\nu'}{2} + B_1\right|\right] \\
\ge\ & -(K+x)\gamma\left|\nu-\nu'\right|\frac{\nu+\nu'}{2} - e^{\frac{\left|\nu^2-\nu'^2\right|}{2}\gamma}\left|\nu'-\nu\right|x\mathbb{E}\left[\left|(\gamma-1)\frac{\nu+\nu'}{2} + \nu + B_1\right|\right].
\end{aligned}$$

Remarking that for $y \in \mathbb{R}$, $\mathbb{E}|y+B_1| = y(2\mathcal{N}(y)-1) + \frac{2e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$ and combining the resulting inequality with the one deduced by exchanging ν and ν', we conclude that Eq. (13) holds. □

A.2. *Proofs of the auxiliary results of Section 4.2*

A.2.1. *Proof of Lemma 4.8*

Proof. Let $\theta > 0$ and $x > c_i(\theta)$. For $a, b \in \mathbb{R}$ and $t \in [0,\theta]$, we define $Y_t^{a,b} = a + \frac{t}{\theta}(b-a) + \Xi_t$ where $(\Xi_s)_{s\in[0,\theta]}$ is a Brownian bridge on $[0,\theta]$ starting and ending at 0. Then $\left(Y_t^{a,b}\right)_{t\in[0,\theta]}$ is a Brownian bridge on $[0,\theta]$ starting at a and ending at b. For $y \ge 0$,

$$\begin{aligned}
&\mathbb{P}\left(\tau = \theta | \bar{S}_\theta^x = y\right) \\
&= \mathbb{P}\left(\forall t \in [0,\theta],\ Y_t^{0,\frac{1}{\sigma}\left(\ln\frac{y}{x} - \left(r-\frac{\sigma^2}{2}\right)\theta\right)} > \frac{1}{\sigma}\left(\ln\frac{c_i(\theta-t)}{x} - \left(r-\frac{\sigma^2}{2}\right)t\right)\right) \\
&= \mathbb{P}\left(\forall t \in [0,\theta],\quad \Xi_t > \frac{1}{\sigma}\left(\ln\frac{c_i(\theta-t)}{x} - \frac{t}{\theta}\ln\frac{y}{x}\right)\right)
\end{aligned}$$

and the monotonicity of $y \mapsto \mathbb{P}\left(\tau = \theta|\bar{S}_\theta^x = y\right)$ easily follows. For $y > K$, this implies

$$\frac{\mathbb{P}(\tau=\theta, \bar{S}_\theta^x \in (K,y))}{\mathbb{P}(\bar{S}_\theta^x \in (K,y))} \le \mathbb{P}(\tau = \theta|\bar{S}_\theta^x = y).$$

Therefore, to prove the second assertion, we only need to check $\mathbb{P}(\tau=\theta, \bar{S}_\theta^x \in (K,y)) > 0$. Let $\eta = \inf\left\{t \ge 0 | \bar{S}_t^x = \frac{y+K}{2}\right\}$. As $\sup_{t\ge0} c_i(t) \le K$, one has

$$\left\{\tau > \eta, \eta < \theta, \forall v \in [0,\theta-\eta]\bar{S}_{\eta+v}^x \in (K,y)\right\} \subset \left\{\tau = \theta, \bar{S}_\theta^x \in (K,y)\right\}.$$

By the strong Markov property and the continuity of the Black–Scholes model, one deduces

$$\begin{aligned}\mathbb{P}(\tau=\theta,\bar{S}^x_\theta\in(K,y)) &\geq \mathbb{E}\left[\mathbf{1}_{\{\tau>\eta,\eta<\theta\}}\mathbb{P}(\forall v\in[0,t],\bar{S}_v^{\frac{y+K}{2}}\in(K,y))\Big|_{t=\theta-\eta}\right]\\ &\geq \mathbb{P}(\tau>\eta,\eta<\theta)\mathbb{P}(\forall v\in[0,\theta],\bar{S}_v^{\frac{y+K}{2}}\in(K,y))\\ &\geq \mathbb{P}\left(\tau=\theta,\bar{S}^x_\theta\geq y\right)\mathbb{P}(\forall v\in[0,\theta],\bar{S}_v^{\frac{y+K}{2}}\in(K,y)).\end{aligned}$$

The last factor in the right-hand-side is positive. By comonotony,

$$\mathbb{P}\left(\tau=\theta,\bar{S}^x_\theta\geq y\right)=\mathbb{E}\left[\mathbb{P}(\tau=\theta|\bar{S}^x_\theta)\mathbf{1}_{\{\bar{S}^x_\theta\geq y\}}\right]\geq\mathbb{P}(\tau=\theta)\,\mathbb{P}\left(\bar{S}^x_\theta\geq y\right).$$

One concludes by remarking that

$$\begin{aligned}&K\mathbb{E}\left[e^{-r\tau}\right]-x+\mathbb{E}\left[e^{-r\theta}\mathbf{1}_{\{\tau=\theta\}}\left(u_i(0,\bar{S}^x_\theta)+\bar{S}^x_\theta-K\right)\right]\\ &\quad=u_i(\theta,x)>K-x\geq K\mathbb{E}\left[e^{-r\tau}\right]-x\end{aligned}$$

implies positivity of $\mathbb{P}(\tau=\theta)$. □

A.2.2. Proof of Lemma 4.9

Proof. Let $\theta,\epsilon>0$, $x\geq 0$ and τ be an optimal stopping time for $u_i(\theta,x)$. Since

$$u_i(\theta,x+\epsilon)\geq\mathbb{E}\left[e^{-r\tau}\left(K-\bar{S}^{x+\epsilon}_\tau\right)^+\mathbf{1}_{\{\tau<\theta\}}+e^{-r\theta}u_i(0,\bar{S}^{x+\epsilon}_\theta)\mathbf{1}_{\{\tau=\theta\}}\right]$$

and $(K-\bar{S}^{x+\epsilon}_\tau)^+-(K-\bar{S}^x_\tau)^+\geq\bar{S}^x_\tau-\bar{S}^{x+\epsilon}_\tau$, we have

$$\begin{aligned}&\frac{u_i(\theta,x+\epsilon)-u_i(\theta,x)}{\epsilon}\\ &\quad\geq\frac{1}{\epsilon}\mathbb{E}\left[e^{-r\tau}\left(\bar{S}^x_\tau-\bar{S}^{x+\epsilon}_\tau\right)\mathbf{1}_{\{\tau<\theta\}}+e^{-r\theta}\left(u_i(0,\bar{S}^{x+\epsilon}_\theta)-u_i(0,\bar{S}^x_\theta)\right)\mathbf{1}_{\{\tau=\theta\}}\right]\\ &\quad=-\mathbb{E}\left[e^{-r\tau}\bar{S}^1_\tau\mathbf{1}_{\{\tau<\theta\}}\right]+\mathbb{E}\left[e^{-r\theta}\bar{S}^1_\theta\frac{u_i(0,\bar{S}^{x+\epsilon}_\theta)-u_i(0,\bar{S}^x_\theta)}{\bar{S}^{x+\epsilon}_\theta-\bar{S}^x_\theta}\mathbf{1}_{\{\tau=\theta\}}\right]\\ &\quad=-\mathbb{Q}\left(\tau<\theta\right)+\mathbb{E}^{\mathbb{Q}}\left[\frac{u_i(0,\bar{S}^{x+\epsilon}_\theta)-u_i(0,\bar{S}^x_\theta)}{\bar{S}^{x+\epsilon}_\theta-\bar{S}^x_\theta}\mathbf{1}_{\{\tau=\theta\}}\right]\\ &\quad=-1+\mathbb{E}^{\mathbb{Q}}\left[\left(1+\frac{u_i(0,\bar{S}^{x+\epsilon}_\theta)-u_i(0,\bar{S}^x_\theta)}{\bar{S}^{x+\epsilon}_\theta-\bar{S}^x_\theta}\right)\mathbf{1}_{\{\tau=\theta\}}\right]\end{aligned}$$

where we used $\bar{S}^x_\theta=x\bar{S}^1_\theta$ for the first equality and $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_\theta}=e^{-r\theta}\bar{S}^1_\theta$ for the second one. The first assertion is deduced by dominated convergence using that, according to Lemma 2.2, $x\mapsto u_i(0,x)$ is 1-Lipschitz and therefore almost surely differentiable.

The smallest optimal stopping time for $u_i(\theta,x+\epsilon)$ is $\tau^\epsilon=\theta\wedge\inf\{t\in[0,\theta]|\bar{S}^{x+\epsilon}_t\leq c_i(\theta-t)\}$. Clearly, $\mathbb{P}$-almost surely, for any $\epsilon>\epsilon'$, $\tau^\epsilon\geq\tau^{\epsilon'}$ and one may define $\overline{\tau}$ as $\lim_{\epsilon\to 0_+}\tau^\epsilon$. Moreover, $\overline{\tau}\geq\tau^\star$ where $\tau^\star$ is the smallest optimal stopping time for $u_i(\theta,x)$. As $(\mathcal{F}_t)_t$ is a right-continuous filtration, $\overline{\tau}$ is a stopping time (cf (4.17) p. 46 of [19]). By optimality of τ^ϵ,

$$u_i(\theta,x+\epsilon)=\mathbb{E}\left[e^{-r\tau^\epsilon}\right]K-(x+\epsilon)+\mathbb{E}\left[e^{-r\theta}\mathbf{1}_{\{\tau^\epsilon=\theta\}}\left(u_i\left(0,\bar{S}^{x+\epsilon}_\theta\right)+\bar{S}^{x+\epsilon}_\theta-K\right)\right].$$

Since $x \mapsto x + u_i(0, x)$ is 1-Lipschitz, one may take the limit $\epsilon \to 0$ in this equality and obtain

$$u_i(\theta, x) = \mathbb{E}\left[e^{-r\overline{\tau}}\right] K - x + \mathbb{E}\left[e^{-r\theta}\mathbf{1}_{\{\overline{\tau}=\theta\}}\left(u_i\left(0, \bar{S}^x_\theta\right) + \bar{S}^x_\theta - K\right)\right],$$

which implies that $\overline{\tau}$ is also an optimal stopping time for $u_i(\theta, x)$.

When $\tau^\epsilon < \theta$, $\bar{S}^x_{\tau^\epsilon} \le \bar{S}^{x+\epsilon}_{\tau^\epsilon} \le K$. Therefore

$$\begin{aligned}
&\frac{u_i(\theta, x+\epsilon) - u_i(\theta, x)}{\epsilon}\\
&\le \frac{1}{\epsilon}\mathbb{E}\left[e^{-r\tau^\epsilon}\left(\bar{S}^x_{\tau^\epsilon} - \bar{S}^{x+\epsilon}_{\tau^\epsilon}\right)\mathbf{1}_{\{\tau^\epsilon<\theta\}} + e^{-r\theta}\left(u_i(0, \bar{S}^{x+\epsilon}_\theta) - u_i(0, \bar{S}^x_\theta)\right)\mathbf{1}_{\{\tau^\epsilon=\theta\}}\right]\\
&= -1 + \mathbb{E}^{\mathbb{Q}}\left[\left(1 + \frac{u_i(0, \bar{S}^{x+\epsilon}_\theta) - u_i(0, \bar{S}^x_\theta)}{\bar{S}^{x+\epsilon}_\theta - \bar{S}^x_\theta}\right)\mathbf{1}_{\{\tau^\epsilon=\theta\}}\right].
\end{aligned}$$

Letting $\epsilon \to 0$ in this inequality, we obtain by dominated convergence $\partial_x u_i(\theta, x) + 1 \le \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\overline{\tau}=\theta\}}\left(1 + \partial_x u_i(0, \bar{S}^x_\theta)\right)\right]$, which concludes the proof. □

A.3. Proofs of the auxiliary results of Section 5

A.3.1. Proof of Lemma 5.2

Proof. Let $\theta > 0$. Using the definition of u_i, Eq. (4), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
u_i(\theta, x) &\ge Ke^{-r\theta} - x + e^{-r\theta}\mathbb{E}\left[u_i(0, \bar{S}^x_\theta) + \bar{S}^x_\theta - K\right]\\
&\ge Ke^{-r\theta} - x + e^{-r\theta}\mathbb{E}\left[\alpha\left|(\bar{S}^x_\theta - c_i(0))^+\right|^\beta\right]\\
&\quad + e^{-r\theta}\mathbb{E}\left[\mathbf{1}_{\{\bar{S}^x_\theta \notin V\}}\left(u_i(0, \bar{S}^x_\theta) + \bar{S}^x_\theta - K - \alpha\left|\left(\bar{S}^x_\theta - c_i(0)\right)^+\right|^\beta\right)\right]\\
&\ge Ke^{-r\theta} - x + e^{-r\theta}\mathbb{E}\left[\alpha\left|(\bar{S}^x_\theta - c_i(0))^+\right|^\beta\right] - e^{-r\theta}\mathbb{E}\left[\mathbf{1}_{\{\bar{S}^x_\theta \notin V\}}\alpha\left|\bar{S}^x_\theta\right|^\beta\right]\\
&\ge Ke^{-r\theta} - x + e^{-r\theta}\mathbb{E}\left[\alpha\left|(\bar{S}^x_\theta - c_i(0))^+\right|^\beta\right]\\
&\quad - \alpha e^{-r\theta}x^\beta e^{\beta\left(r-\frac{\sigma^2}{2}\right)\theta + \beta^2\sigma^2\theta}\sqrt{\mathbb{P}(\bar{S}^x_\theta \notin V)}.
\end{aligned}$$

Let $\epsilon > 0$ be such that $(c_i(0) - 2\epsilon, c_i(0) + 2\epsilon) \subset V$. For $x \in (c_i(0) - \epsilon, c_i(0) + \epsilon)$,

$$\begin{aligned}
\mathbb{P}\left(\bar{S}^x_\theta \notin V\right) &\le \mathbb{P}\left(\bar{S}^x_\theta \notin (x-\epsilon, x+\epsilon)\right)\\
&\le 2\mathcal{N}\left\{\frac{1}{\sigma\sqrt{\theta}}\left(\left(r + \frac{\sigma^2}{2}\right)\theta + \ln\max\left(\frac{x-\epsilon}{x}, \frac{x}{x+\epsilon}\right)\right)\right\}.
\end{aligned}$$

We deduce that

$$u_i(\theta, x) \ge Ke^{-r\theta} - x + e^{-r\theta}\mathbb{E}\left[\alpha\left|(\bar{S}^x_\theta - c_i(0))^+\right|^\beta\right] + o(\theta), \tag{14}$$

where the term $o(\theta)$ is uniform for $x \in (c_i(0) - \epsilon, c_i(0) + \epsilon)$. In order to bound the third term of the right-hand-side from below, we first deal with $\phi(\theta) \overset{\text{def}}{=} \mathbb{E}\left[\left|\left(\bar{S}^1_\theta - 1\right)^+\right|^\beta\right]$. Using the change of variables $z = \sigma\sqrt{\theta}u$ for the second equality, we have

$$\begin{aligned}
\phi(\theta) &= \int_0^{+\infty} z^\beta e^{-\frac{1}{2\sigma^2\theta}\left(\ln(1+z)-\left[r-\frac{\sigma^2}{2}\right]\theta\right)^2} \frac{dz}{\sqrt{2\pi\theta}\sigma(1+z)} \\
&\geq e^{-[\frac{r}{\sigma}-\frac{\sigma}{2}]^2\theta} \int_0^{+\infty} z^\beta e^{-\frac{1}{\sigma^2\theta}\ln^2(1+z)} \frac{dz}{\sqrt{2\pi\theta}\sigma(1+z)} \\
&\geq e^{-[\frac{r}{\sigma}-\frac{\sigma}{2}]^2\theta} \int_0^{+\infty} z^\beta e^{-\frac{z^2}{\sigma^2\theta}} \frac{dz}{\sqrt{2\pi\theta}\sigma(1+z)} \\
&= e^{-[\frac{r}{\sigma}-\frac{\sigma}{2}]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \int_0^{+\infty} \frac{u^\beta e^{-u^2}du}{\sqrt{2\pi}(1+u\sigma\sqrt{\theta})} \\
&\geq e^{-[\frac{r}{\sigma}-\frac{\sigma}{2}]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \int_0^{+\infty} \frac{u^\beta e^{-u^2}}{\sqrt{2\pi}}\left(1-u\sigma\sqrt{\theta}\right)du \\
&= e^{-[\frac{r}{\sigma}-\frac{\sigma}{2}]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \frac{1}{\sqrt{8\pi}}\left[\Gamma\left(\frac{1+\beta}{2}\right)-\sigma\sqrt{\theta}\Gamma\left(\frac{3+\beta}{2}\right)\right] \\
&= e^{-[\frac{r}{\sigma}-\frac{\sigma}{2}]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \frac{1}{\sqrt{8\pi}}\Gamma\left(\frac{1+\beta}{2}\right)\left[1-\sigma\sqrt{\theta}\frac{1+\beta}{2}\right].
\end{aligned}$$

Thus, for $\theta < \frac{1}{\sigma^2(1+\beta)^2}$ and $C = \frac{1}{2}e^{-\frac{(\frac{r}{\sigma}-\frac{\sigma}{2})^2}{\sigma^2(1+\beta)^2}}\frac{\sigma^\beta}{\sqrt{8\pi}}\Gamma\left(\frac{1+\beta}{2}\right)$, one has $\phi(\theta) \geq C\theta^{\frac{\beta}{2}}$.

Let $x < c_i(0)$ and $\tau = \inf\left\{t \geq 0 | \bar{S}^x_t \geq c_i(0)\right\}$. For $\theta < \frac{1}{\sigma^2(1+\beta)^2}$, using the strong Markov property then Formula 2.0.2 p. 223 [3], one has

$$\begin{aligned}
\mathbb{E}\left[\left|\left(\bar{S}^x_\theta - c_i(0)\right)^+\right|^\beta\right] &= |c_i(0)|^\beta\, \mathbb{E}\left[\mathbb{E}\left[\left|\left(\bar{S}^1_{\theta-\tau}-1\right)^+\right|^\beta |\mathcal{F}_\tau\right]\mathbf{1}_{\{\tau<\theta\}}\right] \\
&= |c_i(0)|^\beta\, \mathbb{E}\left[\phi\left(\theta-\tau\right)\mathbf{1}_{\{\tau<\theta\}}\right] \\
&\geq |c_i(0)|^\beta\, C\theta^{\frac{\beta}{2}}\mathbb{E}\left[\left(1-\frac{\tau}{\theta}\right)^{\frac{\beta}{2}}\mathbf{1}_{\{\tau<\theta\}}\right] \\
&\geq |c_i(0)|^\beta\, C\theta^{\frac{\beta}{2}}\frac{1}{\sigma}\ln\frac{c_i(0)}{x}\int_0^\theta\left(1-\frac{t}{\theta}\right)^{\frac{\beta}{2}} \\
&\quad\times\frac{1}{\sqrt{2\pi t^3}}e^{-\frac{1}{2\sigma^2 t}\left(\left(\frac{\sigma^2}{2}-r\right)t+\ln\frac{c_i(0)}{x}\right)^2}dt \\
&\geq |c_i(0)|^\beta\, e^{\frac{1}{2\sigma^2}\left[2\left(\frac{\sigma^2}{2}-r\right)\ln\frac{x}{c_i(0)}-\left(\frac{\sigma^2}{2}-r\right)^2\theta\right]}C\theta^{\frac{\beta}{2}} \\
&\quad\times\underbrace{\frac{1}{\sigma\sqrt{2\pi\theta}}\ln\frac{c_i(0)}{x}\int_0^1(1-u)^{\frac{\beta}{2}}\frac{1}{\sqrt{u^3}}e^{-\frac{1}{2\sigma^2\theta u}\ln^2\frac{c_i(0)}{x}}du}_{:=\psi(\theta,x)}.
\end{aligned}$$

Hence

$$\exists M,\eta>0, \forall(\theta,x)\in(0,\eta)\times(c_i(0)e^{-\sigma\theta^{\frac{1}{3}}}, c_i(0)),$$

$$\mathbb{E}\left[\left|\left(\bar{S}^x_\theta - c_i(0)\right)^+\right|^\beta\right] \geq M\theta^{\frac{\beta}{2}}\psi(\theta, x). \tag{15}$$

Setting $\gamma(x) = \frac{1}{\sigma\sqrt{\theta}}\ln\frac{c_i(0)}{x}$, we have $\psi(\theta, x) = \frac{\gamma(x)}{\sqrt{2\pi}}\int_0^1 (1-u)^{\frac{\beta}{2}} u^{-\frac{3}{2}} e^{-\frac{\gamma^2(x)}{2u}} du$. With the change of variables $t = \frac{1}{u} - 1$, we deduce that $\psi(\theta, x) = \frac{\gamma(x)}{\sqrt{2\pi}} e^{-\frac{\gamma^2(x)}{2}} \Gamma\left(\frac{\beta}{2}+1\right) \mathrm{U}\left(\frac{\beta}{2}+1; \frac{3}{2}; \frac{\gamma^2(x)}{2}\right)$ where $\mathrm{U}(a, b, z) = \frac{1}{\Gamma(a)}\int_0^{+\infty} e^{-tz} t^{a-1}(1+t)^{b-a-1} dt$ is the confluent hypergeometric function of the second kind. By 13.5.2 p. 504 [1],

$$\text{for } z \to +\infty, \quad \mathrm{U}\left(\frac{\beta}{2}+1; \frac{3}{2}; z\right) = z^{-(\frac{\beta}{2}+1)}(1 + O(1/z)).$$

Then we choose θ small enough to ensure that $x(\theta) = c_i(0)e^{-\sigma\sqrt{\theta((2-\beta)|\ln\theta| - (\delta+\beta)\ln|\ln\theta|)}}$ is well defined. Since $\gamma(x(\theta)) = \sqrt{(2-\beta)|\ln\theta| - (\delta+\beta)\ln|\ln\theta|}$ tends to $+\infty$ as $\theta \to 0$, we deduce

$$\begin{aligned}
\psi(\theta, x(\theta)) &= \frac{\Gamma\left(\frac{\beta}{2}+1\right)2^{1+\frac{\beta}{2}}}{((2-\beta)|\ln\theta| - (\delta+\beta)\ln|\ln\theta|)^{\frac{\beta+1}{2}}\sqrt{2\pi}} \\
&\quad \times \theta^{1-\frac{\beta}{2}}|\ln\theta|^{\frac{\delta+\beta}{2}}\left(1 + O\left(\frac{1}{|\ln\theta|}\right)\right) \\
&= \frac{\Gamma\left(\frac{\beta}{2}+1\right)2^{1+\frac{\beta}{2}}}{\sqrt{2\pi}(2-\beta)^{\frac{\beta+1}{2}}}\theta^{1-\frac{\beta}{2}}|\ln\theta|^{\frac{\delta-1}{2}}\left(1 + O\left(\frac{\ln|\ln\theta|}{|\ln\theta|}\right)\right).
\end{aligned}$$

Plugging this into Eq. (15), we conclude that there exists a constant $\kappa > 0$ such that as $\theta \to 0$,

$$\mathbb{E}\left[\left|\left(\bar{S}^{x(\theta)}_\theta - c_i(0)\right)^+\right|^\beta\right] \geq \kappa\theta|\ln\theta|^{\frac{\delta-1}{2}}\left(1 + O\left(\frac{\ln|\ln\theta|}{|\ln\theta|}\right)\right).$$

With Eq. (14), this implies that

$$u_i(\theta, x(\theta)) \geq K - x(\theta) + \theta\left(\kappa|\ln\theta|^{\frac{\delta-1}{2}} - rK\right) + o(\theta)$$

and the conclusion follows by positivity of the factor $\kappa|\ln\theta|^{\frac{\delta-1}{2}} - rK$ for θ small enough. □

A.3.2. Proof of Lemma 5.5

Proof. Ideas are similar to those of the proof of Proposition 2.2 of [10]. For $\alpha > 0$, according to Proposition 4.1, there exists $\eta > 0$ such that $\sup_{w\in[0,\eta]} c_i(w) \leq c_i(0) + \frac{\alpha}{2}$. Let us suppose that $t_1 \in [0, \eta]$. Let $x \leq c_i(0) + \alpha$ and $v \geq 0$.

Setting $\tilde{\tau} = \inf\left\{w \geq 0 | \bar{S}^x_w \geq c_i(0) + \alpha\right\}$, we have

$$\mathbf{1}_{\{\tau\geq v\}} \geq \mathbf{1}_{\left\{\tau\geq\tilde{\tau}, \tilde{\tau}\leq v, \forall w\in[\tilde{\tau}, v],\ \bar{S}^x_w > c_i(0)+\alpha\right\}}.$$

Using the strong Markov property, we deduce that

$$\mathbb{P}(\tau \geq v) \geq \mathbb{P}(\tau \geq \tilde{\tau}, \tilde{\tau} \leq v)\,\mathbb{P}\left(\inf_{w\in[0,v]} \bar{S}^1_w > \frac{c_i(0) + \frac{\alpha}{2}}{\hat{c}_i(0) + \alpha}\right). \tag{16}$$

Whereas, by continuity of the trajectories of $\bar{S}^x$ and since $x \leq c_i(0) + \alpha$,

$$\mathbf{1}_{\{\tau\geq v, \bar{S}^x_v \geq c_i(0)+2\alpha\}} \leq \mathbf{1}_{\{\tau\geq\tilde{\tau}, \tilde{\tau}\leq v,\ \bar{S}^x_v\geq c_i(0)+2\alpha\}}.$$

Again by the strong Markov property, we deduce that

$$\mathbb{E}\left[\left(\bar{S}_v^x\right)^p \mathbf{1}_{\{\tau \geq v,\ \bar{S}_v^x \geq c_i(0)+2\alpha\}}\right] \leq \mathbb{E}\left[\mathbf{1}_{\{\tau \geq \tilde{\tau}, \tilde{\tau} \leq v\}} (c_i(0)+\alpha)^p\, \mathbb{E}\left[\left(\bar{S}_w^1\right)^p \mathbf{1}_{\left\{\bar{S}_w^1 \geq \frac{c_i(0)+2\alpha}{c_i(0)+\alpha}\right\}}\right]_{w=v-\tilde{\tau}}\right]. \tag{17}$$

Then by defining $\tilde{\mathbb{P}}$ as $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{p\sigma B_t - \frac{p^2\sigma^2 t}{2}}$, we get

$$\mathbb{E}\left[\left(\bar{S}_v^x\right)^p \mathbf{1}_{\{\tau \geq v,\ \bar{S}_v^x \geq c_i(0)+2\alpha\}}\right] \leq \mathbb{P}(\tau \geq \tilde{\tau},\ \tilde{\tau} \leq v)\,(c_i(0)+\alpha)^p\, e^{\left(pr+\sigma^2 \frac{p(p-1)}{2}\right)v} \times \sup_{0 \leq w \leq v} \tilde{\mathbb{P}}\left(\bar{S}_w^1 \geq \frac{c_i(0)+2\alpha}{c_i(0)+\alpha}\right). \tag{18}$$

Notice that for any $t, x, y \geq 0$, $\mathbb{P}(\bar{S}_t^x \geq y) \leq \tilde{\mathbb{P}}(\bar{S}_t^x \geq y)$. So, we deduce that

$$\frac{\mathbb{E}\left[\left(1+\left(\bar{S}_v^x\right)^p\right)\mathbf{1}_{\{\tau \geq v, \bar{S}_v^x \geq c_i(0)+2\alpha\}}\right]}{\mathbb{P}(\tau \geq v)} \leq \frac{\left(1+(c_i(0)+\alpha)^p\, e^{\left(pr+\sigma^2\frac{p(p-1)}{2}\right)v}\right)\sup_{0\leq w\leq v}\tilde{\mathbb{P}}\left(\bar{S}_w^1 \geq \frac{c_i(0)+2\alpha}{c_i(0)+\alpha}\right)}{\mathbb{P}\left(\inf_{w\in[0,v]} \bar{S}_w^1 > \frac{c_i(0)+\frac{\alpha}{2}}{\hat{c}_i(0)+\alpha}\right)}. \tag{19}$$

This concludes the proof since when v tends to 0, the numerator tends to 0 whereas the denominator tends to 1. □

A.3.3. Itô–Tanaka formula

Lemma A.1. *For $i \geq 1$, assume that D_i is difference of two convex functions. Then*

$$\begin{aligned} du_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x)) &= \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x))\rho_i'(\bar{S}_t^x)d\bar{S}_t^x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a))dL_t^a(\bar{S}^x)\rho_i''(da) \\ &\quad + \frac{1}{2}\partial_{xx}u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x))\left(\rho_i'(\bar{S}_t^x)\right)^2 d\left\langle \bar{S}^x\right\rangle_t. \end{aligned}$$

Proof. By the Itô–Tanaka formula,

$$d\rho_i(\bar{S}_t^x) = \rho_i'(\bar{S}_t^x)d\bar{S}_t^x + \frac{1}{2}\int_{\mathbb{R}} dL_t^a(\bar{S}^x)\rho_i''(da).$$

Hence $X_t = \rho_i(\bar{S}_t^x)$ is a continuous semi-martingale with bracket $\langle X\rangle_t = \int_0^t \left(\rho_i'(\bar{S}_s^x)\right)^2 d\left\langle \bar{S}^x\right\rangle_s$. By Lemma 3.3, since $\theta_d^{i-1} > 0$, the function $f(x) = \partial_{xx}u_{i-1}\left(\theta_d^{i-1}, \bullet\right)$ is bounded. The next lemma ensures that

$$\begin{aligned} du_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x)) &= \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x))\left(\rho_i'(\bar{S}_t^x)d\bar{S}_t^x + \frac{1}{2}\int_{\mathbb{R}}\rho_i''(da)dL_t^a(\bar{S}^x)\right) \\ &\quad + \frac{1}{2}\partial_{xx}u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x))\left(\rho_i'(\bar{S}_t^x)\right)^2 d\left\langle \bar{S}^x\right\rangle_t. \end{aligned}$$

One concludes since, by Proposition 1.3 p. 222 [19], $\mathbb{P} \otimes |\rho_i''|(da)$ a.e., the measure $dL_t^a(\bar{S}^x)$ is supported by $\{t : \bar{S}_t^x = a\}$. $\square$

Lemma A.2. *Let X be a continuous semi-martingale and f a $\mathcal{C}^1$ function, $\mathcal{C}^2$ on $[0, x^\star)$ and $(x^\star, +\infty)$, such that either $\inf_{x\in\mathbb{R}} f''(x)$ or $\sup_{x\in\mathbb{R}} f''(x)$ is finite. Then, almost surely,*

$$\int_0^t \mathbf{1}_{\{X_s = x^\star\}} d\langle X\rangle_s = 0 \quad \textit{and}$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s .$$

Proof. The first assertion is a consequence of the occupation times formula and ensures that differentiability of f' at $x^\star$ is not needed for the right-hand-side of the second equality to be well defined. By hypothesis, there exists $0 \leq M < \infty$ such that either $x \mapsto f(x) + Mx^2$ or $x \mapsto f(x) - Mx^2$ is convex and consequently f is the difference of two convex functions. So we can apply the Itô–Tanaka formula and conclude by the occupation times formula. $\square$

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